FIBONACCI TRIPLES AND PYTHAGOREAN TRIANGLES OF EQUAL AREA

by

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ABSTRACT

Fibonacci triples (3, 5, 8) form two adjacent Fibonacci pairs (3, 5) and (5, 8) of successive Fibonacci numbers, which are the solutions of diophantine equations of the form $a^2 \pm ab + b^2 = c^2$. The solutions of these Diophantine equations are related to the problem of Pythagorean triangles of equal areas.

Key Words and Phrases

Rational Pythagorean Triangles (abbr. RPT), Fibonacci Triples, n-dimensional Lucas Numbers, Adjacent Fibonacci pairs, Diophantine equations.

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1. Definitions and Previous Results

A triangle with sides $a$, $b$, $c$ which are represented by a triple $(a, b, c)$ of natural numbers will be called a Rational Pythagorean Triangle (abbr. RPT), if and only if there exists $(u, v)$,

$$(u, v) \in \mathbb{N}^2 - \{(0, 1)\}, \quad u > v, \text{ such that } a = u^2 - v^2, b = 2uv, \quad c = u^2 + v^2, \quad a, b, c \in \mathbb{N} - \{0\} = 1, 2, \ldots \quad (1.1)$$

Denote such triangle by $\text{RPT}(u, v)$

The area of the right triangle
(denoted by $S(u, v)$) is $S(u, v) = \frac{1}{2} \ab = uv (u^2 - v^2)$.

The first who asked the question to find triplets RPTs having equal areas was the great Diophantus [4] and Dickson [3] enlarged the topic.

Let $D$ be a triangle with integral sides and $\hat{C} = 120^\circ$ one of its angles.

Then, if $c$ is the side opposite $\hat{C}$ and $a, b$ the two adjacent sides of $\hat{C}$, we have by $c^2 = a^2 + b^2 - 2ab \cos \hat{C}$

$$a^2 + ab + b^2 = c^2$$

$$a + b > c > b > a; \; a, b, c \in \mathbb{N} - \{0\}$$

(1.2)

and if $c = 60^\circ$ we have

$$a^2 - ab + b^2 = c^2$$

$$b > c > a > 0; \; a, b, c \in \mathbb{N} - \{0\}$$

(1.3)

Batela [1] connected the solutions of these two diophantine equations with the areas of the triangles. The totality of solutions to $a^2 + ab + b^2 = c^2$ is given in parameter form by Hasse [5]. No explicite solutions of (1.2) and (1.3) were known.

Since (1.2) and (1.3) are homogeneous diophantine equations, with a proper linear transformation, they can be reduced to a simple diophantine equation which can be solved explicitly.

Theorem 1 of [1] states that if $a, b, c$ satisfy the equation $a^2 + ab + b^2 = c^2$, (where $a + b > c > b > a$, and $a, b, c \in \mathbb{N} - \{0\}$, then the three triangles RPT(c, a), RPT(c, b), and RPT(a + b, c) all have the same area, namely $S = abc(a + b)$.

To find an infinite number of solutions to the equation $a^2 + ab^2 + b^2 = c^2$ (but not necessarily all), let $a = y - 1$, and $b = y + 1$.

The substitution yields $c^2 - 3y^2 = 1$, which is Pell's equation

(1.4)

(1.5)

There are infinitely many pairs of numbers $(u_n, v_n)$ which satisfy Pell's equation $u_n^2 - 2v_n^2 = 1$ where

$$u_n + \sqrt{3} \; v_n = (2 + \sqrt{3})^n, \; n = 0, 1, \ldots$$

(1.6)

From (1.6) we deduce
\[ u_n = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2i} \cdot 2^{n-2i} \cdot 3^i \quad n = 0, 1, \ldots \]

\[ v_n = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} \cdot 2^{n-1-2i} \cdot 3^i \quad n = 0, 1, \ldots \]  

for \( n = 2m \)

\[ u_{2m} = \sum_{i=0}^{m} \binom{2m}{2i} \cdot 2^{2m-2i} \cdot 3^i \quad m = 1, 2, \ldots \]  

\[ v_{2m} = \sum_{i=0}^{m-1} \binom{2m}{2i+1} \cdot 2^{2m-1-2i} \cdot 3^i \quad m = 1, 2, \ldots \]  

\[ (u_0, v_0) = (1, 0) \]

or for \( n = 2m + 1 \)

\[ U_{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{2i} \cdot 2^{2m+1-2i} \cdot 3^i \quad m = 0, 1, \ldots \]  

\[ U_{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{2i+1} \cdot 2^{2m+1-2i} \cdot 3^i \quad m = 0, 1, \ldots \]  

Some of these pairs are \((u_0, v_0) = (1, 0), (u_1, v_1) = (2, 1), (u_2, v_2) = (7, 4), (u_3, v_3) = (26, 15)\), etc.  

Consider the pair \( u_2 = c - 7 \) and \( v_2 = y = 4 \) then \( a = 3, b = 5 \) and \( c = 7 \) and the triangles are

\[ \text{RPT}(7, 2) = (40, 42, 58) \]
\[ \text{RPT}(7, 5) = (24, 70, 74) \]
\[ \text{RPT}(8, 7) = (15, 112, 113) \]
all these triangles (1.11) have area \(3.5\cdot 7.8 = 840\).

Note that the three underlined numbers are three consecutive Fibonacci numbers.

Theorem 2 in [1] states that if \(a, b, c\) satisfy the equation \(a^2 - ab + b^2 = c^2, b > c > a > 0\), then there are the three triangles RPT\((b, c)\), RPT\((c, a)\) and RPT\((a, b-c)\) all have the same area, namely \(S = abc(b - c)\).

Using a similar method to find an infinite number of solutions let
\[
a = \frac{1}{2}(y + 1) \text{ and } b = y - 1; \ (y \geq 2) \ (1.4a).
\]

Substituting yields \(4c^2 - 3(y - 1)^2 = 4\ (y \text{ odd})\). If \(y\) is odd, then \(y - 1\) is even and the equation may be divided by 4 to give:
\[
c^2 - \frac{3(y - 1)^2}{2} = 1 \text{ which is again Pell's equation (1.5a)}
\]

Again the \((u_n, v_n)\) pairs provide infinitely many solutions, where \(u_n = c\) and
\[
v_n = \frac{y - 1}{2} \text{ or } y = 2v_n + 1.
\]

Consider again the pair \((7, 4)\); then \(c = 7\) and \(y = 9\) so \(a = 5, \ b = 8, \ c = 7\, \text{ and the triangles are:}\)
\[
RPT(8, 7) = (15, 112, 113) \\
RPT(7, 5) = (24, 70, 74) \\
RPT(7, 3) = (40, 42, 58)
\]
(1.11a) are the same triples as in (1.11). Now we are asking whether two successive Fibonacci numbers would be solutions \((a, b)\) of (1.2) or (1.3)?

Note that \(c\) does not have to be a Fibonacci number, but that \(a + b\) will be the third Fibonacci number involved in the RPT triplet.

The Fibonacci sequence with \(F_1 = F_2 = 1, \ F_{n-3} = F_n + F_{n+1}\), \(n = 1, 2, \ldots\)
\[\text{goes } 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\]

Here \((F_4, F_5) = (3, 5)\) is a solution of the Diophantine equation (1, 2).

(i. e.) \(3^2 - 3 \cdot 5 + 5^2 = 7^2\) and \((F_5, F_6) = (5, 8)\) is a solution of (1.3)

(i. e.) \(5^2 - 5 \cdot 8 + 8^2 = 7^2\).
In this paper we intend to look for more pairs of adjacent Fibonacci numbers as serving as solutions for (1.2) and (1.3).

2. Main Result

At a conference in April 1990, R. Pinch suggested using \( \phi \) and \( \overline{\phi} \) as the Fibonacci generators for this problem. Using standard notation for Fibonacci numbers:

\[
\begin{align*}
\text{let } \phi &= \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \overline{\phi} &= \frac{1 - \sqrt{5}}{2} \\
\text{where } \phi + \overline{\phi} &= 1 \quad \text{and} \quad \phi \overline{\phi} &= -1.
\end{align*}
\]

(2.1)

Let \( F_n = \frac{1}{\sqrt{5}} (\phi^n - \overline{\phi}^n) \) \hspace{1cm} (2.2)

(\text{So the first Fibonacci number corresponds to } n = 1)

Let \( a = F_n \), \( b = F_{n+1} \), and substitute it into

\[
a^2 + ab + b^2 = c^2, \text{ getting } F_n^2 + F_nF_{n+1} + F_{n+1}^2 = c^2 \quad (2.3)
\]

Substituting the \( \phi \) notation:

\[
\frac{1}{5} \left( \phi^{2n} - 2\phi^n \overline{\phi}^n + \overline{\phi}^{2n} \right) + \frac{1}{5} \left( \phi^{2n+1} - \phi^n \overline{\phi}^n - \phi \overline{\phi}^{n+1} + \overline{\phi}^{2n+1} \right) +
\]

\[
\quad + \frac{1}{5} \left( \phi^{2n+2} - 2\phi^{n+1} \overline{\phi}^{n+1} + \overline{\phi}^{2n+2} \right) = c^2
\]

Factor the \( \frac{1}{5} \), rearrange, and use (2.1) to get:

\[
\frac{1}{5} \left[ \phi^{2n} + \phi^{2n+1} + \phi^{2n+1} + \phi^{2n+2} + \overline{\phi}^{2n+2} + 2(-1)^{n+1} - 2(-1)^n \right] = c^2
\]

Thus

\[
\phi^{2n} + \phi^{2n+1} + \phi^{2n+1} + \phi^{2n+2} + \overline{\phi}^{2n+2} + 1 = c^2 \quad (2.4)
\]
with $-1$ when $n$ is even, $+1$ when $n$ is odd.

Let $L_n$ be a Lucas number of the form $L_n = \phi^n + \overline{\phi}^n$ \hspace{1cm} (2.5)

Then we have (from 2.4)

$\begin{align*}
L_{2n} + L_{2n+1} + L_{2n+2} \pm 1 &= 5c^2 \hspace{1cm} (2.6)
\end{align*}$

By the recursive property at $L_n$, we have $L_{2n} + L_{2n+1} = L_{2n+2}$ \hspace{1cm} (2.7)

or $2L_{2n+2} \pm 1 = 5c^2$ \hspace{1cm} (2.8)

Solving: \hspace{1cm} $L_{2n+2} = \frac{5c^2 \mp 1}{2}$

Use the $\phi$ notation to discover the relation between the Lucas number

$L_n = \phi^n + \overline{\phi}^n$ and $F_n = \frac{1}{\sqrt{5}} (\phi^n - \overline{\phi}^n)$

$L_n^2 = \phi^{2n} + 2 \phi^n \overline{\phi}^n + \overline{\phi}^{2n} = \phi^{2n} + 2(-1)^n + \overline{\phi}^{2n}$

$F_n^2 = \frac{1}{5} (\phi^{2n} - 2\phi^n \overline{\phi}^n + \overline{\phi}^{2n}) = \frac{1}{5} (\phi^{2n} - 2(-1)^n + \overline{\phi}^{2n})$

$L_n^2 = 2(-1)^n = \phi^{2n} + \overline{\phi}^{2n} = 5F_n^2 + 2(-1)^n$

So \hspace{1cm} $L_n^2 = 5F_n^2 \pm 4$ \hspace{1cm} (2.2)

with $+4$ when $n$ is even, $-4$ when $n$ is odd.

By (2.9) we have $L_{2n+2}^2 = 5F_{2n+2}^2 + 4$ (+ since $2n + 2$ is even) and by (2.8)

$\left( \frac{5c^2 \pm 1}{2} \right)^2 = 5F_{2n+2}^2 + 4$

It is fairly easy to see that $5F_{2n+2}^2 + 4$ is always a perfect square, so let
\[ F = \sqrt{5F_{n+2}^2 + 4} \]. Then \( \frac{5c^2 \pm 1}{2} = F \), or solving for \( c^2 \):
\[ c^2 = \frac{2F \pm 1}{5} \]

This equation clearly has infinitely many solutions if \( F \) is allowed to be arbitrary, but for a Fibonacci solution, we must have \( F = \sqrt{5F_{2n+2}^2 + 4} \). So far, the only Fibonacci solution, we have discovered to yield a \( \frac{2F \pm 1}{5} \), which is a perfect square is 55 (when \( n = 4 \), so this gives \( a = 3, b = 5 \)).

Using a direct attack, and calculating \( a^2 + ab + b^2 \) for pairs of consecutive Fibonacci numbers, it can be shown that if \( F_n^2 + F_nF_{n+1} + F_{n+1}^2 \) is a perfect square, then \( n \) must be of the form \( 30m + k \), where \( m \) is any integer \( \geq 0 \) and \( k \) is \( 0, 4, 5, 10, 18, 23, 24, 28 \) or \( 29 \).

(2.10)

This follows from observations about the last three digits of perfect squares which end in 1 or 9 (since \( F_n^2 + F_nF_{n+1} + F_{n+1}^2 \) always ends with a 1, 3, 7 or 9; the actual pattern is 3, 7, 9, 9, 9, 7, 3, 1, 1, 1, 3, 7, etc.).

If \( \ldots abc \) is a perfect square with last three digits \( a, b, \) and \( c \), where \( c \) is 1 or 9, then \( b \) is always even, and the relationship between \( a \) and \( b \) is as follows: If \( a \) is odd, then \( b \) is 2 or 6. If \( a \) is even, then \( b \) is 0, 4, or 8.

The last three digits of \( F_n^2 + F_nF_{n+1} + F_{n+1}^2 \) have a pattern which repeats with a period of 750; the only ones which could be a perfect square \( c^2 \) are the ones with \( n \) as indicated in (2.10).

A computer check up to \( n = 275 \) has not yet revealed any perfect squares, except of course for \( n = 4 \).

Our explorations have not yet yielded any final answer to the question of whether there are any adjacent Fibonacci numbers, other than 3 and 5, which can be used for \( a \) and \( b \) in the solution of the Diophantine equation \( a^2 + ab + b^2 = c^2 \). At this time it appears that there are not. We will continue the search.
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