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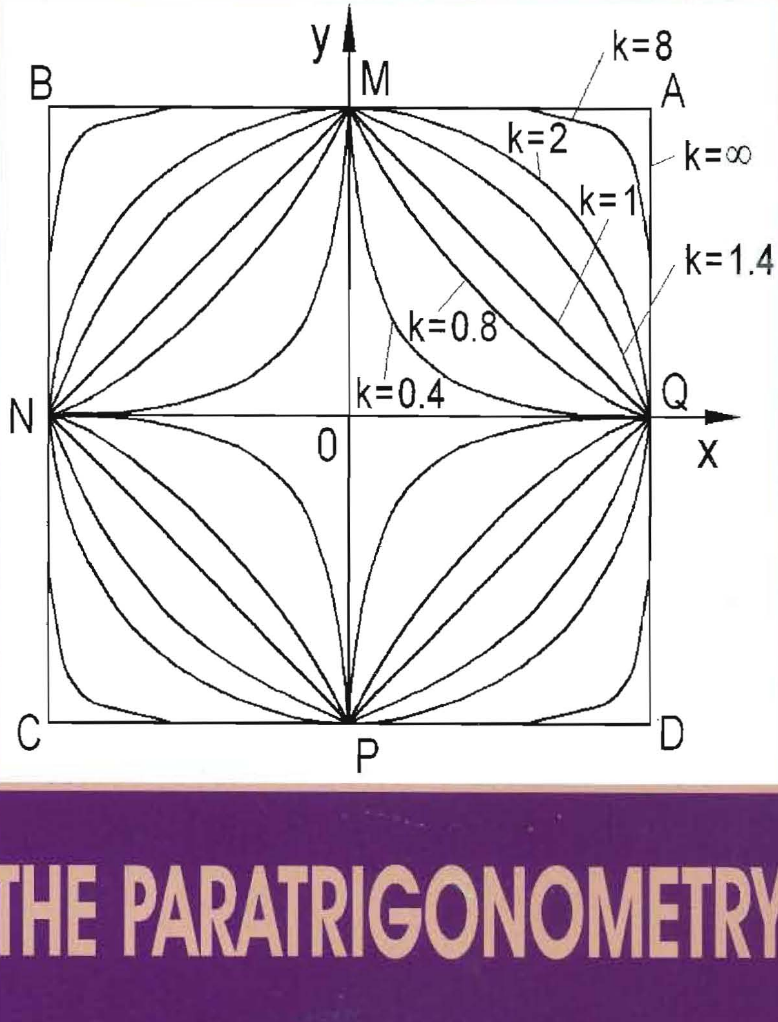
Malvina BAICA
Mircea CÂRDU



THE PARATRIGONOMETRY

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Malvina BAICA   Mircea CÂRDU

THE PARATRIGONOMETRY

Nonconventional trigonometry

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This volume comprises all of the authors' research papers published between 2002 and 2010 in the *Scientific Bulletin of the University "Politehnica" of Timisoara (Romania) – Transaction of Mathematics and Physics*.

They have as their subjects's diverse aspects of non-conventional trigonometry, named by the authors "*Paratrigonometry*".

The Paratrigonometry, having a general feature, contains, as special cases, the Classical Trigonometry and also the Quadratic Trigonometry developed in the years '30 from the passed century by Professor Valeriu Alaci of the University "Politehnica" of Timisoara.

This book is organized in a manner that allows each chapter to be read independently as well as in the context of the assembled chapters. There is some repetition of formulas and figures to facilitate a smoother understanding. We dedicate this book to all who are passionate about new directions in Mathematics, students and specialists alike.

THE ACADEMY OF TECHNICAL
SCIENCES OF ROMANIA

THE UNIVERSITY OF WISCONSIN
WHITEWATER, WI, USA

"POLITEHNICA" UNIVERSITY
OF TIMISOARA, ROMANIA

THE PARATRIGONOMETRY

Nonconventional trigonometry

Prof. **Malvina BAICA** – PhD
Dipl. Eng. **Mircea CÂRDU**

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BIOGRAPHY OF MALVINA BAICA

The wife of Adrian Baica and daughter of Adam (a lawyer) and Cornelia Bunghiu, Malvina Florica Baica was born in Oravita, Romania, on the 3rd of November 1942; After gaining her baccalaureate at *Oravita Lyceum* in 1960 she spent, five years at the *University of Timisoara*, earning a BS in Mathematics and Physics (mechanics) and an European MS in Mathematics (Projective and Differential Geometry). She taught for two years, but subsequently emigrated, with political asylum to the United States. Resuming her studies she was awarded her MA in Mathematics (Algebra and Number Theory) by *Illinois Institute of Technology of Chicago* in 1974 and her Ph.D. in Mathematics (Algebraic Number Theory and Algebra) by the *University of Houston*, Texas, in 1980. Her Ph.D. advisor was Jürgen Schmidt and her Ph.D. dissertation advisors were Helmut Hasse and Leon Bernstein.

Since 1992, Professor of Mathematics and Computer Science at the *University of Wisconsin – Whitewater*, Dr. Baica jointed the academic staff there in 1984 as a lecturer, and was appointed assistant professor the following year. She had previously held positions as assistant professor at *Marquette University* (Wisconsin), *Marshall University* (West Virginia) and *Valparaiso University* (Indiana).

Her research results were published by reputable journals in more than ~~30~~ 80 papers.

Among her publications the most significant are "*An Algorithm in a Complex Field (ACF) and its Applications to the Calculations of Units*" (which is a new and very powerful algorithm, which turned out to be the General Euclidean Algorithm) known as Baica's General Euclidean Algorithm (BGEA); "*Trigonometric Identities*" (now known in mathematical literature as Baica's Trigonometric Identities), where Gauss' two trigonometric identities given by Gauss without proof become particular cases of Baica's Trigonometric Identities proved true in general; "*Putting last Digits first yields Multiples*" (a joint paper with the late Helmut Hasse). Using her (BGEA) algorithm she proved all the famous opes problems in algebraic number theory, including Fermat Last Theorem (FLT).

She has given professional talks at Conferences and other meetings including *the International Congress of Mathematicians*, held at the University of California, Berkley, in 1986, *the International Conference of Number Theory* in Quebec, Canada, in 1987, *the American Mathematical Society's Centennial Celebration in Providence*, Rhode Island, in 1988 and *the International Conferences on Analytical Number Theory* at the University of Illinois, Allerton Park, Urbana, also in 1988. After 1989 (Romanian Revolution) she returned to *Alma Mater* giving more conferences on her favorite subjects in Romania. As consequence she joined *Pannonian Applied Mathematical Meetings (PAMM)*, finding a favorable climate to meet prestigious European mathematicians and scientists for mathematical and interdisciplinary cooperation.

She had received a distinguished professional leadership award and is included in the "*International Directory of Distinguished Leadership*", third edition. Also, she is included in *"Who is Who" International* and in *"Who is Who" for Women.*

BIOGRAPHY OF MIRCEA CÂRDU

Born in Gradinari, Caras-Severin county – Romania, on the 31st October 1934. After gaining his baccalaureate at *Oravita Lyceum General Dragalina* in 1952 he spent five years (until 1957) at the *University "Politehnica" of Timisoara* earning a BS, as dipl. engineer, specialized in Thermic Machines in the frame of *Mechanical Faculty*.

In the period 1957 – 1958 he worked at the *Automotive Reparation Company* (*IRA*) No. 13 from Timisoara as foreman, and in the period 1959 – 1968 at the *Machinebuilding Works from Resita* (*UCMR*), Department of steam turbine design. In the period 1968 – 1994 he worked at the *Researchand Design Institute for Thermopower Equipment* (*ICPET*) from Bucharest, as chief of Design Department for turbo-machines, and general manager, in the period 1979 – 1986 and 1990 – 1994 respectively.

Always was keen on Mathematics which make up the support of a lot of published scientific papers as, for example:

- "*About the computer programming of the steam turbine calculus*" (in Romanian), Energetica Review, No .5 – 6, 1978;
- "*The optimizing of angles* α_1 *of impulse steam turbines nozzles*", Revue Roumain de Sciences Techniques – Serie Electrotechnique et Energetique, 31, No. 2, 1986;
- "*Regarding a global methodology to estimate the energy-ecologic efficiency of thermopower plants*" – co-author: M. Baica, Energy Conversion and Management, Vol. 40, No. 1, 1999.

He was specialized in the field of steam turbines and other thermopower equipment at the following famous companies: *LMZ Sankt Petersburg* from Russia, *Rateau-Schneider* and *Alstom* from France, *Clarck* and *General Electric* from USA, *ENEL* from Italy (at the *AEIA proposal*), *AECL* from Canada.

With the business purposes (export, technical co-operation etc.) he was also in: Germany, Hungary, Poland, Czechoslovakia, Ukraine, Bulgaria, Greece, China, India, Sri-Lanka.

His research results were published by journals from Romania and USA in more than 100 papers. Also, he is the author of 18 conferences and 8 patented inventions.

He is the author of the book "*Paths to the all horizons (From an engineer diary)*", AGIR Publishing House, Bucharest, 1996. Also, is the co-author of the book "*Steam turbines*", Technical Publishing House, Bucharest, 1976.

He had received the following Romanian State medals: "*Work Order*", "*Scientific Merit Order*", "*Order Tudor Vladimirescu*".

He is member of the "*General Association of the Engineers in Romania* (*AGIR*)", the "*Romanian National Comitee of the World Energy Council*" and of the "*Romanian Institute of Energy (IRE)*".

PREFACE

It is a pleasure and also an honor for me, to write some words about this book. One of the authors, Mircea Cârdu, is one of my good fellows in the field of technological development. Mrs. Malvina Baica, in his turn, is known as a distinguished mathematician.

The book is a completion of an idea of Professor Valeriu Alaci. I have had him as professor on Mathematical Analysis at the "*Politehnica*" University of Timişoara. I could remember here the words of Latin poet Ovidiu, expiated at Tomis (at present Constanţa – Romania): "*The end crowns the work*".

The book, with a striking mathematical content, is the work of two authors with various professions, a mathematician and a technological development diploma engineer specialized in the thermal machines field. They have had a mutual purpose and so they constituted a valuable team.

The secret of success of any team consists in a good understanding between his members and in this case the understanding was formed from childhood. The past separated authors, interposing between them an Ocean, but the links was maintained and, after the moment they met again, with the experience and the prestige obtained in both professions, the friendship and cooperation has begun to give results. So, with couple of authors Baica – Cârdu, come out a series of papers with mathematical character. These papers together constitute the content of this book. Also, was come out a series of papers – with the couple of authors Cârdu – Baica – with technical character, having as subject ecological implications of thermo-power plants operation.

A quality of this volume's chapters is their originality, conferred by the local culture inheritance. The Quality will incite the interest first of all of the specialists who desire to promote the new things.

The book demonstrates, once again, the Science hasn't frontiers and is a world good. As such the book is supported by all those which wish to promote the society progress. *The University of Wisconsin – USA* is a clear testimony of this point of view. This scientific and education institution was supported the respective researches and also was fully participated and founding the papers and this book. Also the contribution of the University "*Politehnica*" Timişoara – Romania was very important.

I recommend *the Paratrigonometry,* so entitled by the authors, being a *nonconventional Trigonometry,* especially for the specialists interested in the materials behavior with deviation of ideally state. Also, I recommend that the specialist applies a creative thinking about the characteristic elements of the Paratrigonometry.

Marius PECULEA
Titular Member of the Romanian Academy

INTRODUCTION

This volume comprises all of the authors' research papers published between 2002 and 2010. They have as their subjects's diverse aspects and applications of a non-conventional trigonometry which we have named "*Paratrigonometry*".

Paratrigonometry is a generalization of classical trigonometry which it contains as a special case. It is also includes as a special case the Quadratic Trigonometry as developed by Professor Valeriu Alaci of the *Politehnica* University of Timisoara – Romania in his book *Quadratic Trigonometry,* Graphic Arts Publishing Institute of Timisoara, 1939.

The authors of this volume, both graduates of institutions of higher education in Timisoara, initiated these generalizations in response to and celebration of the exquisite originally of Professor Alaci's work.

The chapters in this volume concerning what we have called "*Paratrigonometry*" were originally published as papers in the *Scientific Bulletin of the Politehnica University of Timisoara – Transaction of Mathematics and Physics.* A list references is included at the end.

This book is organized in a manner that allows each chapter to be read independently as well as in the context of the assembled chapters. There is some repetition of formulas and figures to facilitate a smoother understanding. We dedicate this book to all who are passionate about new directions in Mathematics, students and specialists alike.

The publication of this book is made possible with the integral financial participation of the University of Wisconsin – Whitewater, Wisconsin, USA. For this assistance, we express our gratitude. We also wish to thank Professor John F. Stone, Dean of Graduate Studies and Continuing Education at the University of Wisconsin – Whitewater.

The Authors

CONTENTS

1. ELEMENTS OF POLYGONAL TRIGONOMETRY

1.1. Introduction

More than 60 years ago Professor Valeriu Alaci of "Politehnica" University of Timisoara, Romania developed the "Quadratic Trigonometry".

At that time, in this new chapter of Mathematics, he introduced a "Trigonometry" which is based on a "Trigonometric Square" inscribed in a unit radius circle, as the classical trigonometry is based on a "Trigonometric Circle" of radius one. This "Quadratic Trigonometry" which we denote with (QT), was for the first time presented and published in a book with about 250 pages [1].

Generalizing the idea of the "Trigonometric Square" to the "Trigonometric Polygon", in this chapter we introduce the basic elements of a "Polygonal Trigonometry" which we denote by (PT). At this time we will only introduce the definitions of these new trigonometric functions and establish some basic relationship in this (PT).

1.2. Fundamental relations in (QT)

As we mentioned above, (QT) is based on an inscribed square in a circle of radius one and is positioned in relation to the coordinate axes as they are represented in Figure 1.1.

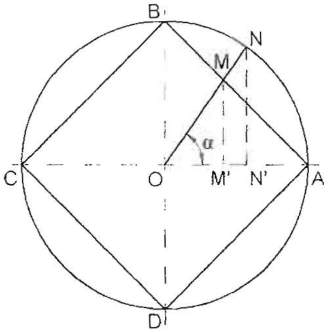

Fig. 1.1. Reference square and its circumscribed circle in the "Quadratic Trigonometry" (QT).

Similar to the trigonometric function in the "Circular Trigonometry" (CT), in (QT), with reference to the trigonometric square $\overline{ABCD}$, we have: $sq\,\alpha = \frac{MM'}{OA}$ and since $OA = 1$ we have $sq\,\alpha = MM'$. We denoted with $sq\,\alpha$ the function "Quadratic Sine" of angle α.

In the same way, for the function "Quadratic Cosine" of angle α we have $cq\,\alpha = OM$. From the triangles $\overline{OMM'}$ and $\overline{ONN'}$ we have a first fundamental relation of (QT), thus:

$$tq\,\alpha = \frac{sq\,\alpha}{cq\,\alpha} = \frac{\sin\alpha}{\cos\alpha} = tg\,\alpha \qquad (1.1)$$

With this relation we make the connection between (QT) and (CT).

We denoted (CT) the "Classical Trigonometry" or "Conventional Trigonometry", which is based on the trigonometric circle with center at 0 and radius one. Thus $OA = OB = OC = OD = 1$, which we also represent in Figure 1.1.

Also in (QT) the functions "Quadratic Secant" and "Quadratic Cosecant" are:

$$\sec q\,\alpha = \frac{1}{cq\,\alpha} \text{ and } \operatorname{cosec} q\,\alpha = \frac{1}{sq\,\alpha}.$$

The second fundamental relation of (QT) results from the right triangles $\overline{OMM'}$ and $\overline{MAM'}$ the last being isosceles with $MM' = AM'$. Since $MM' = sq\,\alpha, OM' = cq\,\alpha$ and $OM' + M'A = 1$ we have:

$$sq\,\alpha + cq\,\alpha = 1 \tag{1.2}$$

It is known that in (CT) the fundamental relations corresponding to (1.2) of (QT) is:

$$\sin^2\alpha + \cos^2\alpha = 1. \tag{1.3}$$

1.3. Polygonal Trigonometry (PT) foundations

(PT) is based on the regular "Trigonometric Polygon" of the number of equal sides n being divisible by four. This trigonometric polygon is inscribed in a circle with radius one. Its position in relation to the reference rectangular axes is represented in Figure 1.2.

Thus the number of sides of the trigonometric polygon is $n = 4 \cdot m$, where m is a positive integer.

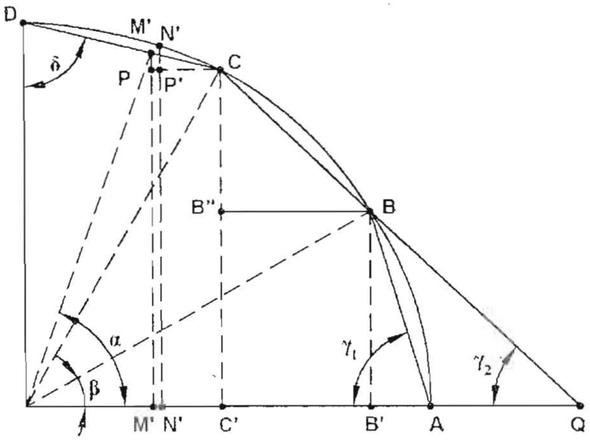

Fig. 1.2. Reference polygon (case $n = 12$), and its circumscribed circle in the "Polygonal Trigonometry" (PT).

For the trigonometric square $n = 4$ and for the trigonometric circle $n = \infty$.

We denote the function "Polygonal Sine" of angle α with $sp(n)\alpha$, and the function "Polygonal Cosine" of angle α with $cp(n)\alpha$, where n represents the number if sides of the trigonometric polygon considered. Thus, using this new notation we can write $sp\,\alpha = sp(4)\alpha$ and $cp\,\alpha = cp(4)\alpha$, respectively for (QT).

Also, we can write $\sin\alpha = sp(\infty)\alpha$ and $\cos\alpha = cp(\infty)\alpha$ respectively for the (CT) case.

Figure 1.2 refers to a trigonometric polygon with twelve sides inscribed in a trigonometric circle with center at 0. To improve the clarity of the figure we represent the first quadrant only $\left(\alpha = 0,\ldots,\frac{\pi}{2}\right)$ of the trigonometric reference.

In this paper we intend to analyze the trigonometric functions variations in the first quadrant only.

1.4. Fundamental relations in (PT)

To obtain relation of general order among the polygonal trigonometric functions we say nothing about the fact that in Figure 1.2 we represented the first quadrant of a trigonometric polygon with twelve sides. In this way instead of $sp(12)\alpha$ and $cp(12)\alpha$ we will use the general notations $sp(n)\alpha$ and $cp(n)\alpha$. We number the circle sectors whose chords are the sides of the considered trigonometric polygon with 1, 2, 3, ..., i where in the trigonometric meaning "i" represents the number of the sectors where the considered angle α opens up. If we refer to the first quadrant we have $i_{\max} = \frac{n}{4}$.

Considering Fig. 1.2 we have

$$sp(n)\alpha = \frac{MM'}{OA} = MM' \text{ and } cp(n)\alpha = \frac{OM'}{OA} = OM'.$$

Also, $tg\,p(n)\alpha = \dfrac{sp(n)\alpha}{cp(n)\alpha}$.

From the similarity of the triangles $\overline{OMM'}$ and $\overline{ONN'}$ we obtain:

$$\frac{sp(n)\alpha}{cp(n)\alpha} = tg\,\alpha\,. \tag{1.4}$$

Relation (1.4) is similar to (1.1) of (QT).

To establish the second fundamental relation of (PT) we see that:

$$OM' + M'C' + C'B' + B'A = 1\,. \tag{1.5}$$

This relation is valid for the situation represented in Fig. 1.2 (Trigonometric Polygon with $n = 12$).

For the situation regarding trigonometric polygons with a larger number of sides, we have a larger number of line segments of category $\overline{C'B'}$ and $\overline{B'A}$ respectively, which represent the projections on the (OA) axis of some integer sides of the trigonometric polygon.

We mention that we denoted with γ_i the angle $\widehat{MCP}$, but we did not mark this angle γ_i on Figure 1.2 because we wish to have a clear picture in Figure 1.2.

We notice that $\beta = \frac{2\pi}{n}$ and from the triangle $\overline{OCQ}$ we have $\gamma_2 = \pi - 2\beta - \delta$. Likewise the angle $\delta = \frac{\pi - \beta}{2} = \frac{\pi}{2} - \frac{\pi}{n}$. Making all the substitutions we obtain $\gamma_2 = \frac{\pi}{2\pi}(n-6)$.

For the general case we have:

$$\gamma_i = (n - 4i + 2)\frac{\pi}{2n}. \tag{1.6}$$

We remember that $OM' = cp(n)\alpha$, and from the triangle $\overline{MCP}$ we have:

$$M'C' = \frac{sp(n\,\alpha) - \sin\left[(i-1)\frac{2\pi}{n}\right]}{tg\,\gamma_i}. \tag{1.7}$$

From triangle $\overline{CBB''}$ we have:

$$C'B' = \frac{\sin\left[(i-1)\frac{2\pi}{n}\right] - \sin\left[(i-2)\frac{2\pi}{n}\right]}{tg\,\gamma_{i-1}}. \tag{1.8}$$

where

$$\gamma_{i-1} = \left[n - 4(i-1) + 2\right]\frac{2\pi}{n}. \tag{1.9}$$

Applying this reasoning further for sectors with order numbers smaller and smaller compared with i and considering relation (1.6) as having a general character, we can deduce the following relation, of general order, between $sp(n)\alpha$ and $cp(n)\alpha$:

$$cp(n)\alpha + \frac{sp(n)\alpha - \sin\left[(i-1)\frac{2\pi}{n}\right]}{tg\,\gamma_i} + \sum_{j=1}^{j=i-1} \frac{\sin\left[(i-j)\frac{2\pi}{n}\right] - \sin\left[(i-j-1)\frac{2\pi}{n}\right]}{tg\,\gamma_{i-j}} = 1. \tag{1.10}$$

where: $i - j \geq 1$.

If in relation (1.10) with the help of (1.9) we perform $n = 4$ (square) we obtain relation (1.2) characteristic of (QT). Consequently, the fundamental relations in (PT) are relations (1.4) and (1.10).

Using these relations we can calculate the values of the trigonometric functions in (PT).

In Figure 1.3 we represented graphically the function $cp(n)\alpha$ (in the first quadrant) for $n = 4$, $n = 8$ and $n = 12$ and for comparison the function $\cos\alpha = cp(\infty)\alpha$.

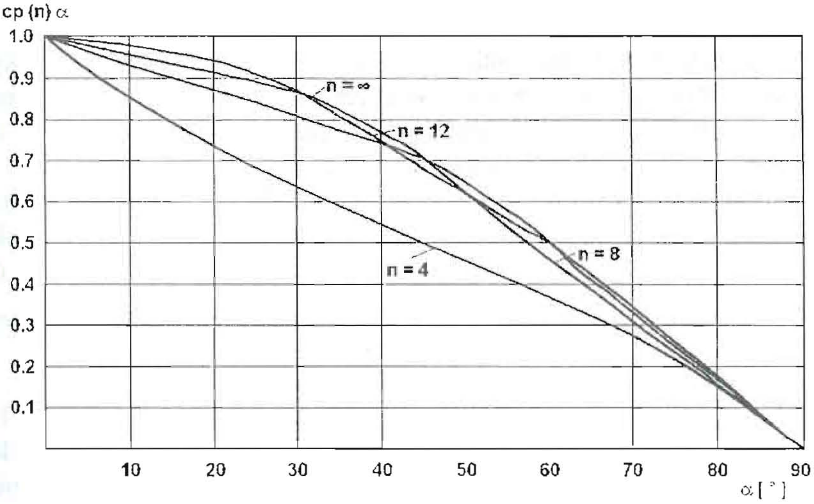

Fig. 1.3. The representation of the functions $cp(n)\alpha$ for $n = 4$, $n = 8$, $n = 12$ and $n = \infty$, in the first quadrant $\left(\alpha = 0, \ldots, \dfrac{\pi}{2}\right)$.

We see that on the points which mark the contact on the circumscribed circle with the trigonometric basic polygon (see Figure 1.2) we have $cp(n)\alpha = \cos\alpha$.

1.5. Conclusions of Chapter 1

"Quadratic Trigonometry" (QT), elaborated by the Romanian Professor Valeriu Alaci is based on the "Trigonometric Square" as well as the "Classical Trigonometry" (CT) is based on the "trigonometric Circle".

In this chapter 1 we elaborated the basic elements of the "Polygonal Trigonometry" (PT), which was developed by using a reference geometrical figure, any regular polygon whose number of sides is a multiple of 4. In this way (QT) become particular case of (PT) when the number of sides of the trigonometric polygon is the minimum possible.

2. REGARDING THE GENERAL CHARACTER OF THE POLYGONAL TRIGONOMETRY

2.1. Introduction

In the chapter 1 the authors presented the basics of the polygonal trigonometry (PT) together with the fundamental relations of PT starting from the foundations of the quadratic trigonometry (QT) elaborated by Professor V. Alaci of the University "Politehnica" of Timisoara, Romania, in the 1930's.

Distinct from the classical trigonometry (CT) which is based on the trigonometric circle with center at O and radius one, QT is developed on a trigonometric square inscribed in a circle with $r = 1$, having its corner at the angles, 0 (zero), $\frac{\Pi}{2}$, Π, $\frac{3\Pi}{2}$, 2Π expressed in radians (respectively 0°, 90°, 180°, 270° and 360° expressed in degrees) of the trigonometric circle.

Similarly, the fundamental relations of PT presented in the chapter 1, were established based on a regular trigonometric polygon inscribed in a circle with $r = 1$. In order to maintain the symmetry conditions this trigonometric polygon must have a number of sides equal to a multiple of four, thus

$$n = 4 \cdot m \tag{2.1}$$

where n is the number of sides of the trigonometric polygon, and m is a positive integer.

Just as for trigonometric square, the corners of the trigonometric polygon after $n/4$, $n/2$ and 3 $n/4$ sides are situated at the angles 0, $\frac{\Pi}{2}$, Π, $\frac{3\Pi}{2}$, of the circle with $r = 1$ circumscribed about the trigonometric polygon.

We make the observation that the trigonometric square is a trigonometric polygon with the minimum possible number of sides ($n = 4$ implies $m = 1$).

Referring to Figure 2.1 where we represent the first quadrant of a trigonometric polygon with 12 sides, $n = 12$ and thus $m = 3$, in the chapter 1 we established the following fundamental relation in PT, valid also for a trigonometric polygon with any number of sides n (respecting $n = 4 \cdot m$):

$$cp(n)\alpha + \frac{sp(n)\alpha - \sin(i-1)\frac{2\Pi}{n}}{tg\gamma_i} + \sum_{j=1}^{j=i-1} \frac{\sin\left[(i-j)\frac{2\Pi}{n} - \sin(i-j-1)\frac{2\Pi}{n}\right]}{tg\gamma_{i-j}} = 1 \tag{2.2}$$

where $cp(n)\alpha$ is the polygonal (p) cosines of α function for the trigonometric polygon with *n* sides, $sp(n)\alpha$ is the polygonal (p) sines of a function respectively, *i* is the current number of the circular sector where angle α is situated, counted in the trigonometric direction starting from $\alpha = 0$, and

$$\gamma_i = (n - 4 \cdot i + 2)\frac{\Pi}{2n} \tag{2.3}$$

$$\gamma_{i-j} = [n + 4 \cdot (i - j) + 2]. \tag{2.4}$$

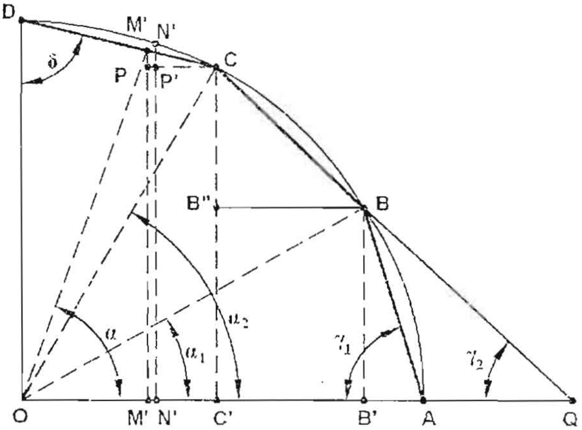

Fig. 2.1. Reference polygon (case $n = 12$) and the circumscribes circle in the Polygonal Trigonometry.

Another important relation valid in all CT, QT and PT is the one which defines the tangent of α function:

$$tg\,\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{sq\,\alpha}{cq\,\alpha} = \frac{sp(n)\alpha}{cp(n)\alpha} \tag{2.5}$$

where $sq(\alpha)$ and $cq(\alpha)$ are quadratic sines and quadratic cosines of a functions respectively. Thus it follows:

$$tp(n)\alpha = tq(\alpha) = tg\,\alpha. \tag{2.6}$$

In the chapter 1 we showed that applying formula (2.2) in the trigonometric square ($n = 4$) case we obtain the fundamental relation of QT, namely

$$sq\,\alpha + cq\,\alpha = 1. \tag{2.7}$$

In what follows, this chapter will analyze the manner in which the basic elements of PT are applied in the CT case, which in fact represents a particular case of PT (when $n = \infty$), as well as in the QT case which represents the other extremes value of n ($n = 4$).

Evidently, $n_{\min} = 4$ and $n_{\max} = \infty$.

2.2. Classical Trigonometry (CT) a particular limiting case of the Polygonal Trigonometry (PT)

For the above mentioned analysis we start using Figure 2.1 and the geometric elements from this figure, with the help of which we obtain the fundamental relation (2.2) applied in the PT. Thus, in the Figure 2.1 we see that for the triangles $OB'B$ and $OC'C$ having the sharp corners B and C of the trigonometric polygon situated on the trigonometric circle, the Pythagorean Theorem gives the relations:

$$\left(\overline{OB'}\right)^2+\left(\overline{BB'}\right)^2=1 \tag{2.8}$$

$$\left(\overline{OC'}\right)^2+\left(\overline{C'C}\right)^2=1\,. \tag{2.9}$$

In the CT (with reference to the trigonometric circle) we have $\overline{OB'}=\cos\alpha_1$ and $\overline{B'B}=\sin\alpha_1$ and respectively, $\overline{OC'}=\cos\alpha_2$ and $\overline{C'C}=\sin\alpha_2$ and thus:

$$\cos^2\alpha_1+\sin^2\alpha_1=1 \tag{2.10}$$

$$\cos^2\alpha_2+\sin^2\alpha_2=1\,. \tag{2.11}$$

But the trigonometric circle is a trigonometric polygon with an infinite number of sides.

Thus all the points which form this circle could be considered as sharp corners (as well as B and C) of the trigonometric polygon with $n=\infty$. In other words, in the trigonometric circle case, applied to CT, the relations (2.8) and (2.9) and respectively (2.10) and (2.11) are valid for any point on the circle. Therefore, regarding the current angle α, we have:

$$\cos^2\alpha+\sin^2\alpha=1\,. \tag{2.12}$$

This (2.12) is a fundamental relations of CT.

It follows then that the trigonometric circle represents the upper limit (for $n=\infty$) of the trigonometric polygon. Consequently, CT represents a particular case (for $n=\infty$) of the PT.

2.3. The general character of the Polygonal Trigonometry (PT)

From the previous section of this chapter and from chapter 1 it follows that PT is generally applicable; the mathematical elements which guide us to its fundamental relation make it valid for both QT and CT.

QT is situated at the lower (inferior) limit of the number of sides of the trigonometric polygon ($n=4$), and CT represents the upper (superior) limit of the PT from this point of view ($n=\infty$).

The fundamental relations of PT have a general character and those of QT and CT represent particular cases of the PT, and they are as follows:

- QT – relation (2.7)
- PT – relation (2.2)
- CT – relation (2.12)

The relations (2.7.) and (2.12) can also be written as follows:

$$\cos^k \alpha + \sin^k \alpha = 1 \tag{2.13}$$

where $k = 1$ for QT and respectively $k = 2$ for CT.

Considering the above mentioned facts, logically it appears that the formula (2.13) could also be valid in the PT, the exponent value k varying in the closed interval $(1 \leq k \leq 2)$ and depending on the value of n which characterizes the corresponding trigonometric polygon.

The investigate the validity of such a hypothesis we proceeded to calculated the values of k as a function of the angle α for three distinct values of n from the range $4 < n < \infty$, these values being $n = 8$, $n = 16$ and $n = 24$.

For this reason we use formulas (2.6) and (2.13) and we have:

$$\left[cp(n)\alpha\right]^k = \frac{1}{1+(tg\,\alpha)^k} \tag{2.14}$$

k is part of the formula (2.14) and as such in order to determinate its values we give k different values in the domain $1 \leq k \leq 2$. From equation (2.14) using logarithms we obtain the following relation for $cp(\alpha)$:

$$cp(n)\alpha = e^z \tag{2.15}$$

where

$$z = \frac{\ln R}{k} \tag{2.16}$$

$$R = \frac{1}{1+(tg\,\alpha)^k}. \tag{2.17}$$

Value of $cp(n)\alpha$ resulting from formula (2.15) is then compared with the one calculated with the exact relation for this function resulting from formula (2.2) where $sp(n)\alpha$ is replaced with the right side term of the equality

$$sp(n)\alpha = \left[cp(n)\alpha\right] \times tg\,\alpha. \tag{2.18}$$

With successive trials we obtain an exact value for k. In this way we find that k depends not only on n but also on the value of the angle α.

The variation of k as a function of α, for these three values of n, namely $n = 8$, $n = 16$ and $n = 24$, is graphically represented in Figure 2.2.

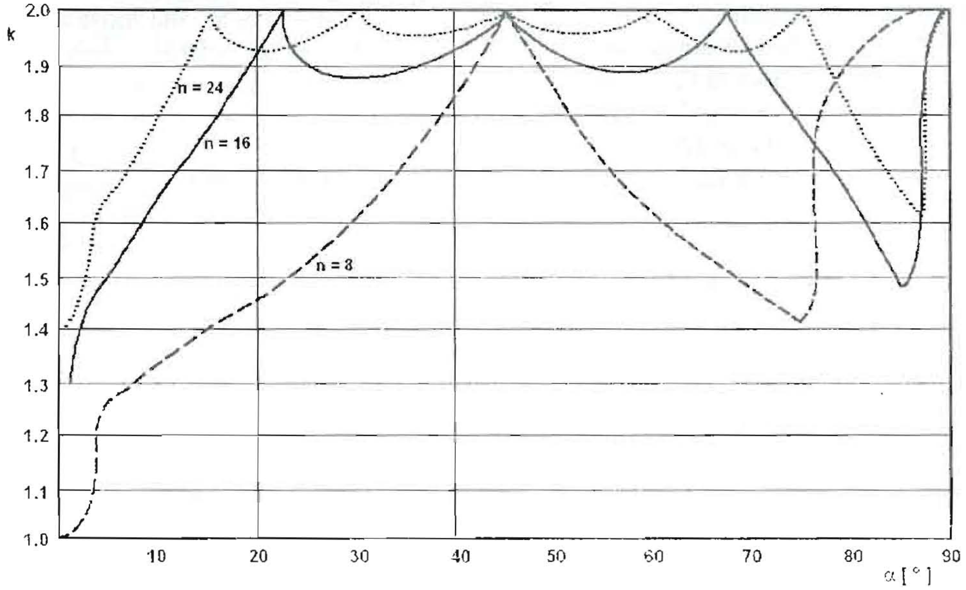

Fig. 2.2. The variation of the exponent k as a function of the angle α for three values of n (in the first trigonometric quadrant).

We calculated the average values of k for all values of the angle a in the first quadrant of the trigonometric circle. The values thus obtained for k_a in these three analyzed cases ($n = 8$, $n = 16$ and $n = 24$) are contained in Table 2.1, where we give synthetically the general character of PT.

Table 2.1

Synthesis of the general character of Polygonal Trigonometry

n	4	8	16	24	...	∞
The Trigonometry	QT		PT			CT
Fundamental relations	(1.7)		(1.2)			(2.5)
	(1.5)		(1.5)			(1.5)
		$k_a =$	$k_a =$	$k_a =$		
k	4	(1.673)	(1.822)	(1.901)	...	2

In Table 2.1, the values of k_a were given in parentheses since they can not be practically used in formula (2.13) which for $4 < n < \infty$ is proved to be only hypothetical.

The value of k is constant for any value of α, only for $n = 4$ ($k = 1$) and $n = \infty$ ($k = 2$).

In Figure 2.3 we represent the values of k (for $n = 4$) – point A – and k_a for $n = 8$, $n = 16$ and $n = 24$ (points B, C, D), and the curve which unites these points.

This curve was traced with an interrupted line, since it is not continuous.

It only units some single points (A, B, C, D, ...) characterized by abscissa values $n = 4 \cdot m$, where m is a positive integer, as we said at the beginning.

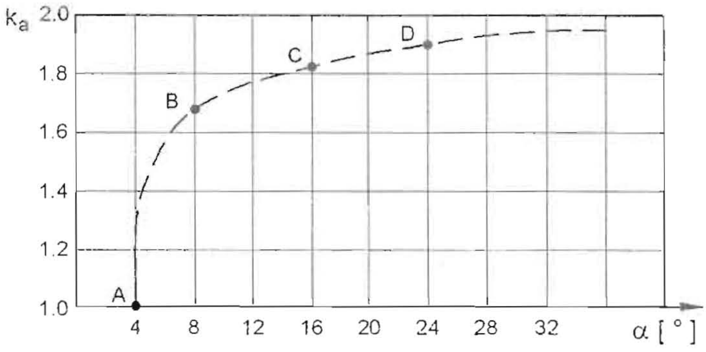

Fig. 2.3. The average value variation of the exponent k as a function of n.

The curve ABCD ... of Figure 2.3 is interesting because it reveals a sharp increase of k_a from $n = 4$ to $n = 8$, after which k_a increases slowly with the increase of n, tending asymptotically toward $k = 2$ for $n = \infty$.

2.4. Conclusions of Chapter 2

The Polygonal Trigonometry (PT) whose basic elements were presented in Chapter 1 has a general character such that the Quadratic Trigonometry (QT) [1] and the Classical Trigonometry (CT) represent the limiting cases for PT.

QT is derived from the PT, when the number of sides n of the trigonometric polygon has a minimum value $n = 4$, and CP is derived from the PT when n has the maximum value $n = \infty$.

The fundamental relation of the QT is (2.7) and in the CT the fundamental relation is (2.12) both of these relations are of the form of relation (2.13) such as

$$\cos^k \alpha + \sin^k \alpha = 1 .$$

In the extreme cases of the value of n, the k exponent has constant values as a function of the angle α. Thus, for $n = 4$ (QT), $k = 1$, and for $n = \infty$ (CT), k = 2.

For all the other cases ($4 < n < \infty$), the values of k are functions of both n and α. The k exponent varies in these cases, as a function of n and α in the interval $4 < n < \infty$, the k exponent attains the value $k = 2$ at the sharp corners of the trigonometric polygon (see Figure 2.1) that is, for values of the angle α (in radians) which equal $\left(\frac{2\Pi}{n}\right) \cdot i$, where i is the current number of circle sectors determined by the trigonometric polygon.

The average values of k, denoted by k_a, which do not have a practical application, show its tendency to increase with the increase of n, from $k = 1$ (for $n = 4$) to $k = 2$ (for $n = \infty$).

3. PERIODIC TRANSTRIGONOMETRIC FUNCTIONS

3.1. Introduction

It is known that many phenomenons in Physics and respectively in technical domains have an oscillation character. In many cases these phenomenons can be mathematically modeled with the help of the trigonometric functions $\sin\alpha$ and $\cos\alpha$ respectively. Examples in this content are the unamortized mechanical vibrations [23] acoustic oscillations, electromagnetic waves etc.

There are some oscillation phenomenons of which mathematical representation does not have a sinusoidal form. In their analysis using the Classical Trigonometry (CT), we apply the decomposition of these functions in Fourier series in order to do the mathematical modeling needed. Let give a single example in this regard, concerning line currents for the electrical transformer with a free current [15]. Intensity variation of such current as a function of the period ωt is represented in Figure 3.1.

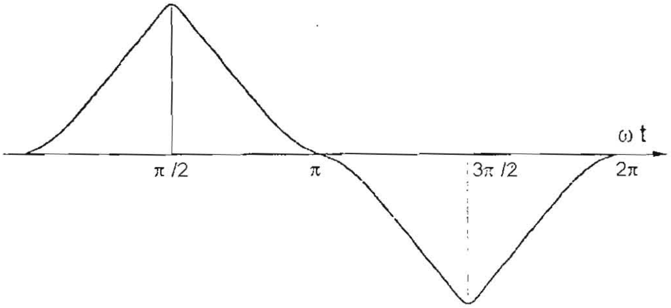

Fig. 3.1. Line current "i" for the electrical transformer with free current, as a function of the temporal period "ωt".

On the other side, in Chapter 1 and 2 the authors analyzed the bases of the Polygonal Trigonometry (PT) using the extended characteristic elements of QT [1].

3.2. Two essential relations in the Transtrigonometry

As it is known the basic relations of CT and QT are the following:

– In CT:

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad (3.1)$$

– In QT [1]:

$$sq\,\alpha + cq\,\alpha = 1 \tag{3.2}$$

– In PT [2, 4]:

$$sq^k\,\alpha + cq^k\,\alpha = 1 \tag{3.3}$$

where k has a variable value included in the domain $1 < k < 2$ [4].

It can be seen that the relations (3.2) and (3.3) are variable for the first trigonometric quadrant ($0 \le \alpha \le \pi / 2$). In order that these relations to be valid for all the four quadrants, they must be written under the form:

$$|sq\,\alpha| + |cq\,\alpha| = 1 \tag{3.4}$$

$$|sq\,\alpha|^k + |cq\,\alpha|^k = 1. \tag{3.5}$$

Relation (3.1) can be kept as it is since its availability from the algebraic point of view is preserved also for the negative values of $\sin\alpha$ and $\cos\alpha$ because they are raised to the second power.

On the basis of relations (3.1), (3.2) and (3.3) from above there appears in a logical way the idea to analyze some periodic functions of type $\sin\alpha$, $\cos\alpha$ of CT which should satisfy a similar relation as (3.3) where k would have a constant value (not a variable on as in PT) and which should be included in the domain $1 < k < 2$. At the lower neighborhood of this domain ($k = 1$) we have QT, and in the upper neighborhood ($k = 2$) we have CT.

We named Transtrigonometry (TT) the chapter of the Trigonometry which includes the domain between QT ($k = 1$) and CT ($k = 2$) and thus it is characterized by k = ct., having values in the domain $1 < k < 2$. The functions of type "sinus α" respectively "cosinus α" we name them "Transtrigonometric sinus α" and "Transtrigonometric cosinus α" and we denote them with $st\,\alpha$ and $ct\,\alpha$. In this way, similarly with the relations (3.4) and (3.5) we will have the relation

$$st\,\alpha^k + ct\,\alpha^k = 1. \tag{3.6}$$

Since k can have any value in the domain $1 < k < 2$ in order to make distinctions between so many situations corresponding the functions $st\,\alpha$, $ct\,\alpha$ etc. by their "order" established by the value of k. The order will be denoted as index to $st\,\alpha$, $ct\,\alpha$ etc. as $st_k\alpha$, $ct_k\alpha$ etc.

Thus, to avoid confusions, relation (3.6) will be written as:

$$st_k\alpha^k + ct_k\alpha^k = 1. \tag{3.7}$$

It can be seen that the relation (3.7) has a general character, and relations (3.4) and (3.5) represent particular cases of that one. Thus, we can write $\sin\alpha = st_2\alpha$, $\cos\alpha = ct_2\alpha$ and $sq\,\alpha = st_1\alpha$, $cq\,\alpha = ct_1\alpha$. In other words, the basic trigonometric functions of QT represent the respective functions "of first order" in TT. The same functions of CT represent the respective functions "of second order" of TT. Thus, the relation (3.7) is the first essential relation of TT.

On the other side, as we have shown in the Chapter 1 and Chapter 2, we can easily prove that the function "tangent" can be included in TT case in the equality:

$$tg\,t_k = tq\,\alpha = tg\,\alpha\,. \tag{3.8}$$

The relation (3.8) represents the second essential relation of TT.

3.3. The characteristics of transtrigonometric functions

For the essential transtrigonometric functions, from (3.7) and (3.8) result the following expressions:

$$st_k\,\alpha = \pm\left(1 + ctg\,\alpha^k\right)^{-1/k} \tag{3.9}$$

$$ct_k\,\alpha = \pm\left(1 + tg\,\alpha^k\right)^{-1/k} \tag{3.10}$$

With relations (3.9) and (3.10) and knowing from CT the values for $tg\,\alpha$ and respectively $ctg\,\alpha$ we can compute the values for the functions $st_k\,\alpha$ and $ct_k\,\alpha$ as functions of angle α, for diverse values of "order" *k*. The signs + (plus) and – (minus) in the front of formulas (3.9) and (3.10) – right sides – are given in function of the quadrant where angle α is situated. As in the CT, for α situated in I and II quadrants, $st_k\,\alpha$ has positive values ad for α in III and IV quadrants, $st_k\,\alpha$ has negative values. On the other side, the function $ct_k\,\alpha$ has positive values in the quadrants I and IV, and negative values in the quadrants II and III.

In Figure 3.2 the function $st_k\,\alpha$ is represented for values of the angle α (expressed in radians) in the domain $0 \le \alpha \le 2\pi$, and for $k = 1$ (QT), $k = 2$ (CT) and $k = 1.4$ (TT). We see that for $k = 2$ the function "sinus" is represented by the classical sinusoid and for $1 \le k \le 2$ the sinusoid curves have forms of "Arabian Archivolt" showing "fractures" for $\alpha = \pi/2$ and $\alpha = 3\pi/2$.

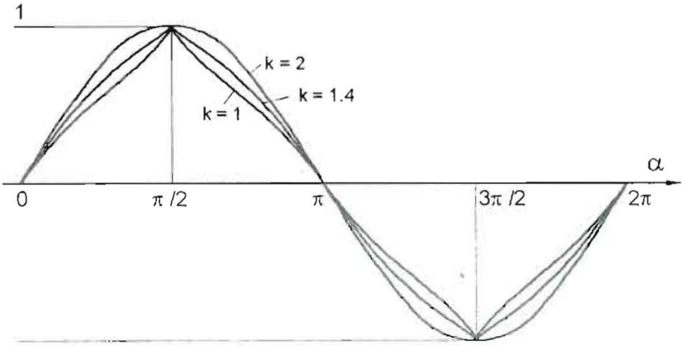

Fig. 3.2. The trigonometric functions $st_k\alpha$ (of order "*k*") for the values of $k = 1$, $k = 1.4$ and $k = 2$.

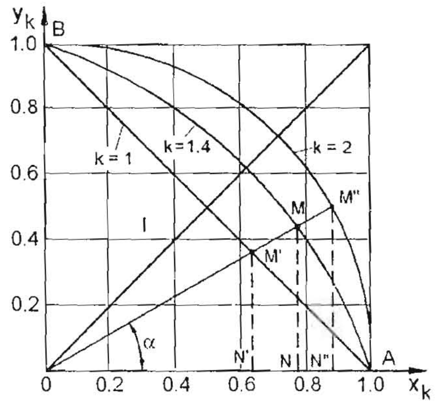

Fig. 3.3. The essential trigonometric figures in CT, QT and TT, in quadrant I (for $k = 1$, $k = 1.4$ and $k = 2$).

In Mathematics there is a distinction between the functions represented by monotonous curves and functions of type $st_k\,\alpha$ (for $k = 1$ and $k = 1.4$) represented by broken graphs consisting of several smooth arcs [20].

If in CT the functions $\sin\alpha$ and $\cos\alpha$ can be illustrated under geometrical form by referring to the "Trigonometric Circle" having a radius equal with the unity ($R = 1$), in QT this is done for $sq\,\alpha$ and $cq\,\alpha$, by referring to the "trigonometric Square" [1] or better said "Trigonometric Rhombus" if we regard its position referring with the two orthogonal axis (horizontal and vertical). In its turn, the trigonometric rhombus is inscribed in a circle of $R = 1$.

If we ask the question to determined the form of the Basic Geometric Figure equivalent with the Trigonometric Circle of CT and of the Rhombus (with straight sides) of QT, we see that this in x_k and y_k coordinates is represented by relation (3.7) making $st_k\,\alpha = x_k$ and $ct_k\,\alpha = y_k$. Thus we will have:

$$y_k^k + x_k^k = 1 \tag{3.11}$$

and making y_k explicit we have:

$$y_k = \pm\left(1 + x_k^k\right)^{1/k}. \tag{3.12}$$

If we represent $y_k = f\left(x_k\right)$ in the domain $0 \le \alpha \le \pi/2$ (Quadrant I) for TT of order $k = 1.4$, we will get the curve represented in Figure 3.3. It has the form of a "Curved Rhombus" (a rhombus with curved sides) and obviously, it is also inscribed in a circle of $R = 1$.

In Figure 3.3, for comparison, we represented also the rhombus with straight sides ($k = 1$) characteristic to QT and as much as the classical trigonometric circle ($k = 2$) characteristic to CT, in quadrant I.

These last two figures also result in an analytic form if in relation (3.12) we introduce $k = 1$ and $k = 2$, respectively. The complete trigonometric rhombus, for $k = 1.4$ is represented in Figure 3.4. The values for y_k will be positive (sign +)

in relation $y_k = f(x_k)$ in I and II quadrants, and negative (sign –) in relation (3.11) for quadrants III and IV.

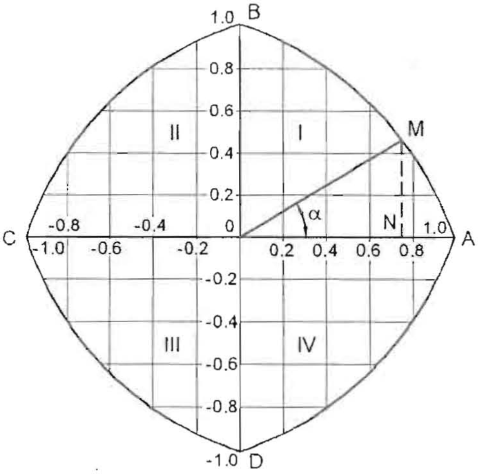

Fig. 3.4. The essential complete trigonometric figure (Trigonometric Rhombus with curved sides), for $k = 1.4$.

In Figure 3.3 we also find the elements with which we can easily repeat the proof for the relation (3.8). Thus, for similar right triangles $OM'N'$, ONM and $OM''N''$, we can write:

$$M'N'/ON' = M''N''/ON'' = MN/ON. \quad (3.13)$$

The trigonometric expression of relation (3.13) is represented by relation (3.8).

3.4. Conclusions of Chapter 3

Transtrigonometry (TT) is a part of Trigonometry in which we study the periodic functions of type *sinus* α, *cosinus* α, etc. which satisfy an essential relation more general then the relation (3.1) valid in the Classical Trigonometry (CT) and respectively the relation (3.4) valid in the Quadratic Trigonometry (QT).

In the TT the essential relation is the relation (3.7) where $k =$ ct., comprised in the domain $1 < k < 2$. CT and QT are particular cases of TT.

The relation (3.8) is common for all three chapters of the Trigonometry (CT, QT and TT). The graphical form of the functions $\sin\alpha$ and $\cos\alpha$ (of CT) is a known continuous curve (the classical sinusoid), and the graphical form of the respective functions in QT and TT are "inscribed" in the classical sinusoid and they are different from it especially because of their "fractures".

Geometrically, these functions are characterized by "Arabian Archivolt" form.

Alike with the "Trigonometric Circle" of CT, in QT there is "Trigonometric Rhombus" with straight sides, and in TT the essential geometric figure is the "Trigonometric Rhombus" with curved sides.

Considering this aspect too, QT and CT represent limiting cases of TT.

4. THE INFRATRIGONOMETRY, AN INFERIOR ORDER NEIGHBORHOOD DOMAIN OF THE TRANSTRIGONOMETRY

4.1. Introduction

It Chapter 3 we established the basic formulas for the relations between the principal trigonometric functions, in a case of a mathematical chapter named Transtrigonometry (TT). These formulas are the following:

$$|st_k\alpha|^k + |ct_k\alpha|^k = 1 \tag{4.1}$$

$$tgt_k\,\alpha = tgq\,\alpha = tg\,\alpha \tag{4.2}$$

where $st_k\,\alpha$ is "transtrigonometric sine" (of order *k*) of an angle α, $ct_k\,\alpha$ is "transtrigonometric cosine" (of order *k*) of the angle α, $tgt_k\,\alpha = st_k\,\alpha / ct_k\,\alpha$ is "transtrigonometric tangent" (of order *k*) of an angle α, $tgq\,\alpha = sq\,\alpha / cq\,\alpha$ is "quadratic tangent" of the angle α (see Quadratic Trigonometry QT [1]) and $tg\,\alpha = \sin\alpha / \cos\alpha$ is the tangent of the angle α of the Classical Trigonometry (CT).

Formulas (4.1) and (4.2), above were established on the basic principles analyzed in the Chapter 1 and Chapter 2.

In TT, the value *k* (named "the order of the trigonometric function [3]) is within the domain $1 < k < 2$. The value *k* = 1 is characteristic to the QT, and *k* = 2 is characteristic to the CT. Next, we will analyze the domain $0 \le k \le 1$ which will generically named Infratrigonometry (IT), being adjacent to TT in the inferior value zone of *k* = 1 (QT).

4.2. The characteristics of the infratrigonometric functions

For the values domain of the order *k* mentioned above ($0 \le k < 1$) we named the basic trigonometric functions as "infratrigonometric sine of order *k* at the angle α", denoted $si_k\alpha$ and respectively "infratrigonometric cosine of order *k* at the angle α", denoted $ci_k\alpha$.

In this case, the formulas (4.1) and (4.2) become:

$$|si_k\alpha|^k + |ci_k\alpha|^k = 1 \tag{4.3}$$

and

$$tgi_k\ \alpha = tgt_k\ \alpha = tq\ \alpha = tg\ \alpha\ . \tag{4.4}$$

Starting from these formulas, as in [3], we obtain the formulas to calculate the values for $si_k\alpha$ and $ci_k\alpha$ as function of $tg\,\alpha$, and respectively $ctg\,\alpha$ of CT. thus we have:

$$si_k\ \alpha = \pm\left[1/\left(1+\left|ctg\ \alpha\right|^k\right)\right]^{1/k} \tag{4.5}$$

and

$$ci_k\ \alpha = \pm\left[1/\left(1+\left|tg\ \alpha\right|^k\right)\right]^{1/k} \tag{4.6}$$

The formulas (4.5) and (4.6) applied in IT are similarly with the corresponding formulas of TT (Chapter 3), mentioning that:

- for TT, $1 < k < 2$;
- for IT, $0 \le k < 1$.

Recall that in QT case, $k = 1$, and in CT case, $k = 2$.

In Figure 4.1 we represent graphically “the sine curve” of the function $si_k\alpha$ for $k = 0.4$, and $k = 0.8$ (both in the IT domain) and, for comparison, classical sine curve, $\sin\alpha$. Evidently, in this case, $k = 2$ (CT).

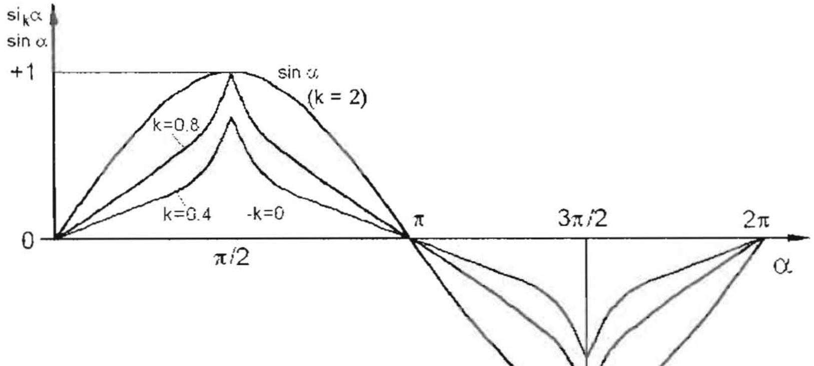

Fig. 4.1. The sine curves for $k = 0$, $k = 0.4$, $k = 0.8$ and $k = 2$.

In Figure 4.1, we also represented the graph of the sine curve for $si_k\alpha$ for $k = 0$ case. This sine curve in fact consists of equal unit length line segments (for absolute value), as we will show detailed in the following chapter. These vertical line segments are periodically situated at the angle α intervals of $\Delta\alpha = \pi$.

We can see that the function $si_k\alpha$ curve has a more pronounced character of an “Arabian Archivolt” than these of the $st_k\alpha$ function (Chapter 3).

The base trigonometric figures represented in the Cartesian coordinate's axis in IT are the same "trigonometric rhombuses with curved sides" as in TT. The distinction consists from the fact that in IT the sides of basis rhombuses are concave in the opposite direction to the reference point O (the coordinate Ox – Oy axis center), while in TT the corresponding sides are concave in the direction to the reference O. remembering that in QT, where $k = 1$, the basis rhombus has the sides (equal sides and equal angles between them) of the line segments form.

Similarly with what we had in Chapter 3 for TT, the basic trigonometric figure in IT is represented by the function:

$$y_k = \pm\left(1-\left|x_k\right|^k\right)^{1/k}. \tag{4.7}$$

In our case (IT), evidently, k has values in the domain $0 \le k < 1$.

In Figure 4.2 we represent the basic "rhombus" with curved sides for $k = 0.4$ and, for comparison, rhombus (with straight sides) for $k = 1$ (QT) and the trigonometric circle for $k = 2$ (CT).

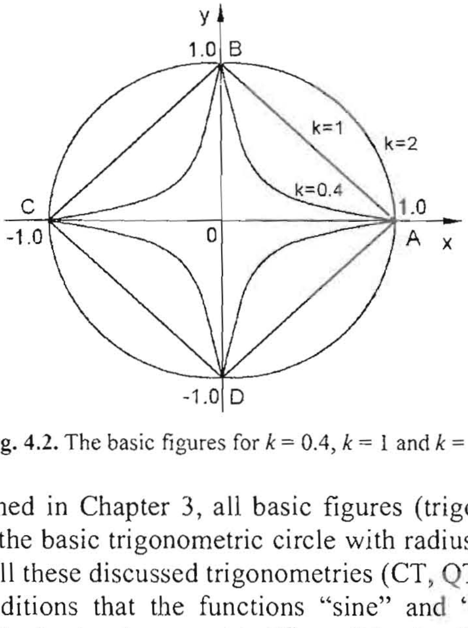

Fig. 4.2. The basic figures for $k = 0.4$, $k = 1$ and $k = 2$.

As we mentioned in Chapter 3, all basic figures (trigonometric rhombuses etc.) are inscribed in the basic trigonometric circle with radius $R = 1$, characteristic for CT. In this way, all these discussed trigonometries (CT, QT, TT, IT) maintained the fundamental conditions that the functions "sine" and "cosine" have values between -1 and $+1$. The basic trigonometric "figure" for $k = 0$ (IT) will be attended in the next chapter.

4.3. The discussion of a special "at limit" case when $k = 0$

The case $k = 0$ represented a limit situation for the values of this parameters in its domain which characterize "the order" of the infratrigonometric functions namely $0 \le k < 1$.

On the other side, k = 1 case (QT) represents the "border" between IT and TT.

Thus, introducing $k = 0$ in formula (4.5) the following situations appear:

4.3.1. When $0<\alpha<(\pi/2)$, $ctg\,\alpha$ has a finite value and having $1/k=\infty$ we have $si_o\alpha=0$.

4.3.2. When $\alpha=0$, we have the second limit situation (beside the first one determined by k = 0). Thus, having $ctg\,0=\infty$, the second term of the denominator in formula (4.5) becomes ∞° which represents one of indetermination cases. In order to eliminate this indetermination we apply the method to calculate the "superior" limit and respectively the "inferior" limits [20] or of "right" limit $\left(\lim\limits_{\alpha\to0^{+}}\right)$ and respectively "left" limit $\left(\lim\limits_{\alpha\to0^{-}}\right)$ as they are named in some professional papers [18].

In our case, as much as for $\alpha\to0^{+}$, and for $\alpha\to0^{-}$, the values of $\alpha\to0^{-}$, the values of $|ctg\,\alpha|$ are finite and we have, as in 4.3.1 above, superior limit $si_0\alpha=0$ [20]. With this value $si_0\alpha$, for the basic relation (2.1) will give us $ci_0\alpha=1$.

4.3.3. When $\alpha=\pi/2$ we have again a limit situation. This time we argue as before, for $ci_0\alpha$ function, thus we will use the basic formula (4.6). We have $ci_0(\pi/2)=0$ and using formula (4.3), $si_0(\pi/2)=1$.

Conform 4.3.1, 4.3.2 and 4.3.3 from above, in Figure 4.1 we represented function $si_0\alpha$, also. Its graph consists from a succession of horizontal line (overlapped on the abscissa), and the vertical line with the absolute value $|1|$, situated at intervals $\Delta\alpha=\pi$ starting from $\alpha=\pi/2$. For $ci_k\alpha$, as in CT, the graph is similar with that for $si_k\alpha$, but sliding with $\pi/2$.

Regarding the basic trigonometric figure of the mathematical model from formula (4.7) we mention that when k = 0, for any value $|x|>0$ we have $|x|^{k}=|x|^{0}=1$ and we have $y=0$. In other words, for any value $|x|>0$, the basic geometrical figure consists in a line which overlapped on the abscissas axis. For $|x|=0$ since in formula (4.7) we have $|x|=0^{\circ}$, the value of y is indeterminate. Trying to eliminate the indetermination does not bring us to any result and thus, for $|x|=0$, y can have any value. This means that $|x|=0$ determine a second line of the basic trigonometric figure which this time overlapped on the ordinates axis.

The same reasoning can be made interchanging x with y in formula (4.7) considering the symmetry of formula (4.3).

We must mention that the basic trigonometrical figure defined above, which is inscribed in the circle with $R = 1$ (of CT) is extended as follows:

– on the abscissa axis $-1 \leq x \leq +1$
– on the ordinates $-1 \leq y \leq +1$

In this way we can see that the form is a cross with all four arms equal and each having the unit value (one). In fact, the cross is the essentialized shape of a rhombus.

Regarding Figure 4.2, it is about the line segments OA; OC on the abscissa, and respectively OB; OD on the ordinate.

According to the definition given in the dictionary *a geometric figure is that one in which the curves and surfaces have simple geometric properties*. Under this aspect, the cross is not a proper geometrical figure, but represents the very well known symbol of the Christian Religion.

4.4. Conclusions of Chapter 4

The Infratrigonometry (IT) represents a chapter of the Trigonometry on which there are studied the basic relations established in Transtrigonometry (TT) (Chapter 3), for values of the order k in an adjacent zone to the characteristic zone of TT ($1 < k < 2$) namely for values of k in the domain $0 \leq k < 1$.

In this way the quadratic Trigonometry (QT), characterized by $k = 1$, represents the border between IT and TT.

The graphs representing the functions $si_k\,\alpha$ and $ci_k\,\alpha$ have forms which are successions of "Arabian Archivolts" as in TT case. In IT the respective archivolts are sharper than in TT case. The sharp character is much stronger if the value of k is smaller.

At the limit, for $k = 0$, the archivolt is transforming in a succession of horizontal and vertical line segments. Thus for $si_0\alpha$ in $0 \leq \alpha < \pi / 2$ interval we have $si_0\alpha = 0$. Also, for $\pi / 2 < \alpha < \pi$ etc. For α having values $\pi / 2$, $3\pi / 2$ etc., $si_k\alpha$ have successively $+1; -1$ values etc. (see Figure 4.1).

The basic trigonometric figure in IT, for $0 < k < 1$ is also a rhombus with curved sides as in TT (see Figure 4.2). The distinction between these two situations (IT and TT) is that in IT the concavity of the curved sides is oriented in an opposite sense to the reference point O (the origin of the coordinate axis), while in TT the respective concavity is oriented to the reference O. For the special case, when $k = 0$, the basic trigonometric figure becomes the cross OA – OC – OB – OD represented in Figure 4.2.

5. THE ULTRATRIGONOMETRY, A SUPERIOR ORDER ADJACENT DOMAIN OF THE TRANSTRIGONOMETRY

5.1. Introduction

From Chapter 3 and 4 we know that the basic formulas for TT and respectively IT are the following:

– for TT ($1 < k < 2$)

$$|st_k\alpha|^k + |ct_k\alpha|^k = 1 \tag{5.1}$$

where $st_k\alpha$ is "sine transtrigonometric of order k of the angle α" and $ct_k\alpha$ is "cosine transtrigonometric of order k of the angle α".

– for IT ($0 \leq k < 1$)

$$|si_k\alpha|^k + |ci_k\alpha|^k = 1 \tag{5.2}$$

where $si_k\alpha$ is "sine infratrigonometric of order k of the angle α" and $ci_k\alpha$ is "cosine infratrigonometric of order k of the angle α".

Another basis formula, common to all previously described trigonometries (TT, IT, CT and QT) [1] is that one which represents the equality of the "tangent" functions for all above cases namely:

$$tgt_k\,\alpha = tgi_k\,\alpha = tgq\,\alpha = tg\,\alpha \tag{5.3}$$

where *tgt* is referred to TT, *tgi* is referred to IT and *tgq* is referred to Quadratic Trigonometry (QT) [1]; $tg\,\alpha$ is the tangent function of the Classical Trigonometry (CT).

Recall that $k = 1$ is characteristic to QT, and $k = 2$ is characteristic to CT. QT is the border between TT and IT and, as we will see, CT is the border between TT and UT.

5.2. The characteristics of ultratrigonometric functions

Similar to formulas (5.1), (5.2) and (5.3), for UT case (for $1 < k \leq \infty$) we have the formulas:

$$|su_k\alpha|^k + |cu_k\alpha|^k = 1 \tag{5.4}$$

$$tgu_k\,\alpha = tg\,\alpha \tag{5.5}$$

where $su_k\alpha$ is "sine ultratrigonometric of order k of the angle α", $cu_k\alpha$ is "cosine ultratrigonometric of order k of the angle α" and $tgu_k\alpha$ is "tangent ultratrigonometric of order k of the angle α". Thus, as in TT and IT cases, starting with formulas (5.4) and (5.5) from above, we obtain the following $su_k\alpha$ and $cu_k\alpha$ in UT:

$$su_k\,\alpha = \pm\left[1/\left(1+|ctg\,\alpha|^k\right)\right]^{1/k} \tag{5.6}$$

and

$$cu_k\,\alpha = \pm\left[1/\left(1+|tg\,\alpha|^k\right)\right]^{1/k}. \tag{5.7}$$

The distinction between formulas (5.6) and (5.7) compared with the corresponding values in TT and IT consists only in the domain values of order k.

Similarly with what we had established for TT and IT regarding the basic trigonometric figures, in UT we have valid the following formula:

$$y_k = \pm\left(1-|x_k|^k\right)^{1/k} \tag{5.8}$$

where k has values in the domain $2 < k < \infty$.

Based on formulas (5.4) and (5.5) we construct the graphs for $su_k\alpha$ having $k = 3$, $k = 8$ and $k = \infty$. For comparison, we give the graph on $\sin\alpha$ function in CT, characterized by $k = 2$, as we have shown. These were represented in Figure 5.1.

We mention that for the clarity of figure (the curves for the values of k given above are very close) the "sine" functions were represented for the domain $0 < \alpha < \pi/2$ only (first trigonometric quadrant).

If in TT and IT the curves of the functions $st_k\alpha$ and $si_k\alpha$ showed fragments for $\alpha = \pi/2$ and carried on at equal intervals with π, in Figure 5.1 we see that the respective curves are monotonous for $k = 3$, $k = 8$ and this is generally valid for $2 < k < \infty$. The form of curve representing $su_k\alpha$ for $k = \infty$ which in its turn has "segments" and explanations are connected with what we have to say in the next chapter.

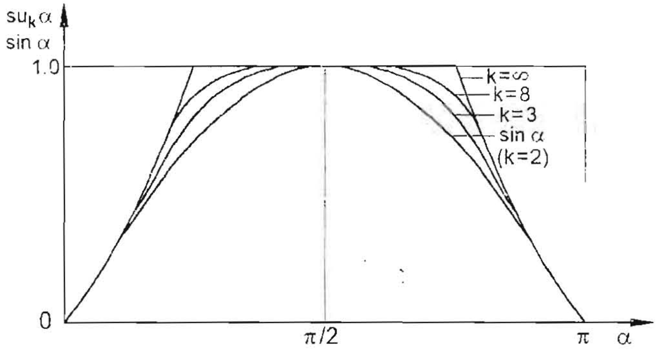

Fig. 5.1. The trigonometric function $su_k\alpha$ for the values of $k = 3$, $k = 8$ and $k = \infty$ and, for comparison, for $k = 2$ (CT).

We can se that the curves for $su_k\alpha$ functions for $2<k<\infty$ have some prominences (in comparison with the curve of $\sin\alpha$ function) round about the value $\alpha=\pi/2$.. These prominences are well-marked when k has a large value. They go to the maximum together with the fragmentation of the respective curve, for $k=\infty$.

The basic trigonometric figures in UT, given by formula (5.8) are represented in Figure 5.2.

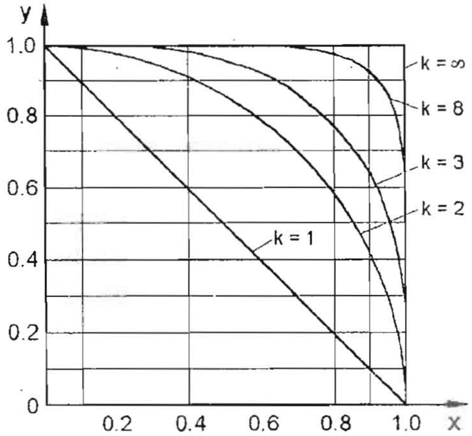

Fig. 5.2. The basic trigonometric figures in UT (for the first trigonometric quadrant) for $k=3$, $k=8$ and $k=\infty$ and, for comparison, for $k=2$ (trigonometric circle of CT).

Again for clarity reasons these figures are represented for the first trigonometric quadrant only. They are referred to the values of the order $k=3$, $k=8$ and $k=\infty$. For comparison reason, in the Figure 5.2 we also represent ¼ of the trigonometric circle in CT, characterized by $k=2$ as well as one of the side of trigonometric rhombus characterized by $k=1$ in QT.

We see that, as in TT case, the basic trigonometric figures for $2<k<\infty$ have the concavity oriented towards the reference O (the coordinate axis origin). The case $k=\infty$ will be discussed in the next chapter.

5.3. The discussion of a special "at limit" case when $k=\infty$

In Chapter 4 we discussed a special "at limit" case when $k=0$. Now, we will discuss the limit case $k=\infty$ which is applied in UT. For this reason, in formula (5.6) we replace $ctg\,\alpha=1/tg\,\alpha$ and obtain:

$$su_k\,\alpha=\pm\left[|tg\,\alpha|/\left(1+|tg\,\alpha|^k\right)^{1/k}\right]. \qquad (5.9)$$

If we introduce in the denominator of formula (5.9) $k=\infty$ and $\alpha=0$ we get into an indetermination situation which can not be solved applying L'Hopital rule.

In this case we proceed to calculate the superior limit and respectively inferior limit [20] of $su_k\alpha$ function, more precisely of the denominator in formula (5.9) for $k=\infty$ and $\alpha=0$. In this way we have:

$$\lim_{\alpha\to 0^+}\left(1+|tg\,\alpha|^{\infty}\right)^0=(1+\Delta)^0 \tag{5.10}$$

where Δ is very small, but yet $\Delta\neq 0$ and thus $(1+\Delta)^0=1$.

Similarly, we have

$$\lim_{\alpha\to 0^-}\left(1+|tg\,\alpha|^{\infty}\right)^0=(1+\Delta)^0. \tag{5.11}$$

For formula (5.11) we also apply, further, the same reasoning like in formula (5.10). Since both $\lim\limits_{\alpha\to 0^+}\varphi$ and $\lim\limits_{\alpha\to 0^-}\varphi=1$ where $\varphi=\left[1+\left|tg\left(0^{\pm}\right)\right|^{\infty}\right]^0$, we also have, by Chapter 4 and [20], $\lim\limits_{\alpha\to 0}\left(1+|tg\,\alpha|^{\infty}\right)^0=1$. Since at the numerator $tg\,\alpha=tg\,0=0$, we have $su_{\infty}0=0$.

For the situation when the angle α has values in the domain $0<\alpha<\pi/4$, we have $0<tg\,\alpha<1$ and thus $su_{\infty}\alpha=tg\,\alpha$.

Carrying on, in order analyzing $su_{\infty}\alpha$ function when $\alpha\geq\pi/4$ we need to return to formula (5.6). In this way we first deal with "at limit" case for $\alpha=\pi/4$ applying the method to compute superior limit and inferior limit for the denominator $\left(1+|ctg\,\alpha|^{\infty}\right)^0$ of formula (5.6) we have $\lim\limits_{\alpha\to(\pi/4)^+}\left(1+|ctg\,\alpha|^{\infty}\right)^0=1$ as $\left|ctg(\pi/4)^+\right|<\infty$. Also $\lim\limits_{\alpha\to(\pi/4)^-}\left(1+|ctg\,\alpha|^{\infty}\right)^0=1$.

Thus, as we shown above, we have $\lim\limits_{\alpha\to(\pi/4)}\left(1+|ctg\,\alpha|^{\infty}\right)^0=1$ and therefore $su_{\infty}(\pi/4)=1$.

In the domain $\pi/4<\alpha<\pi/2$ we have $0<|ctg\,\alpha|<1$ and therefore $0<|ctg\,\alpha|^{\infty}<1$. Consequently, we have $1+|ctg\,\alpha|^{\infty}\neq 1$ and $su_{\infty}\alpha=1$ (having $1/k=1/\infty=0$.

For the limit case, when $\alpha=\pi/2$, we proceed as above, applying $\lim\limits_{\alpha\to(\pi/2)^+}$ and $\lim\limits_{\alpha\to(\pi/2)^-}$ to the denominator of formula (5.6) namely $\left(1+|ctg\,\alpha|^{\infty}\right)^0$. Again we will obtain, and in this case, also $su_\infty(\pi/2)=1$. When α has larger values than $\pi/2$, namely in the domain $\pi/2<\alpha<\pi$, then $su_k\alpha$ function will have the form which can be found in Figure 5.1 as it happen with the periodic function "sine" in general. Certainly, also for larger values of α this fact is similar, $su_k\alpha$ function successively having negative values (in the domain $\pi<\alpha<2\pi$) and again positive etc.

The function $cu_k\alpha$ – see formula (5.7) – as we know from CT, it is in fact represented again by a "sinusoid" (in our case, of type $su_k\alpha$) but it is shifted by $\Delta\alpha=\pi/2$.

Regarding the basic trigonometric figures (Figure 5.2) mathematically modeled by formula (5.3), $k=\infty$ situation is considered again as a "limit case" and is treated similarly as the case which we have discussed before. Thus, for $x<1$ we also have $|x|^\infty<1$ and therefore $y=1$; this represents the horizontal line (parallel with Ox axis) which includes the segment AB of Figure 5.3.

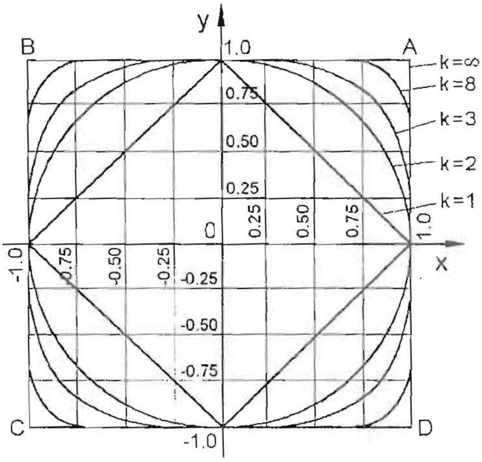

Fig. 5.3. The complete basic trigonometric figures in UT, for $k=3$, $k=8$, $k=\infty$ and, for comparison, for $k=2$ (CT) and $k=1$ (QT).

In this figure, like in Figure 5.2 we represented the basic trigonometric figures completely (in all four trigonometric quadrants) for $k=3$, $k=8$ and $k=\infty$ (UT) and for comparison $k=2$ (CT) and $k=1$ (QT).

"At limit" situation appear for $x=0$. Considering formula (5.8), if we calculate $\lim_{x\to0^+} y$ and $\lim_{x\to0^-} y$ we obtain, for both situations, the value 1. This means that the segment AB of Figure 5.3, mentioned above, completes itself with the point of coordinates (0; 1) (see Figure 5.3).

Everything from above are valid for the situation when in front of formula (5.8) the sign + (plus) is taken into consideration. For the situation when we consider the sign – (minus), everything is referred to the line segment CD of Figure 5.3.

If in formula 5.8 we solve for x as a function of y we obtain:

$$x_k = \pm\left(1+\left|y_k^{\,k}\right|\right)^{1/k} . \tag{5.12}$$

Similarly as above regarding y function where $y=\varphi(x)$ in formula (5.8), we proceed also for the case of function x where $x=\varphi(y)$ in formula (5.12), and obtain the line segments BC and DA. These, together with AB and CD segments form the basic trigonometric figure in UT (the square ABCD) for $k=\infty$.

5.4. Conclusions of Chapter 5

In Ultratrigonometry (UT) we developed the basic relations established in the Transtrigonometry (TT) (see Chapter 3) for the values of the order k comprised in the domain $2<k\leq\infty$ (in TT, $0\leq k<1$).

Thus the classical trigonometry (CT) characterized by $k=2$, represents the border between TT and UT. The function "sine" of UT denoted by $su_k\,\alpha$, for $k<\infty$, graphically works similar with the function $\sin\alpha$ (of CT), but remarkably outside of $\sin\alpha$ graph, round about the value $\alpha=\pi/2$. This prominence is larger when the value of k is larger (Figure 5.1). In any case, the curves representing function $su_k\,\alpha$ (for $k<\infty$) have a monotonous variation, not having "fragments" as it happen in IT and TT cases (inclusive at the borders between them, in QT). When $k=\infty$, in UT appear "fragment" points of the curve which illustrate $su_k\,\alpha$ function. If we consider the domain $0\leq\alpha\leq\pi$, we find these points when $\alpha=\pi/2$ and $\alpha=3\pi/2$ respectively. For $0\leq\alpha<\pi/2$ the function has the same shape as $tg\,\alpha$. We have the same for $(3\pi/2)<\alpha<2\pi$. In the interval $\pi/2\leq\alpha\leq3\pi/2$ we have $su_k\,\alpha=1$ (in fact $su_\infty\,\alpha=1$).

The basic trigonometric figures in UT are "squares" with curved sides as we can see in Figure 5.2 and Figure 5.3. In $k=\infty$ case, the basic trigonometric figure is a real square ABCD as in Figure 5.3. When the value of k decreases from $k=\infty$ to $k=2$ (CT) the curvature of the square ABCD sides becomes more prominent until it is a circle ($k=2$). This circle has now the radius $R=1$ like in CT.

6. THE PARATRIGONOMETRY AND SOME OF ITS SPECIFIC SYMMETRIES

6.1. Introduction

In the Chapter 3, 4 and 5 there were analyzed the "trigonometric" functions which are based on two fundamental relations common to all types of introduced trigonometries as such:

- Transtrigonometry (TT);
- Infratrigonometry (IT);
- Ultratrigonometry (UT).

These two fundamental relations could be written, in a general form, as:

$$|spr_k \alpha|^k + |cpr_k \alpha|^k = 1 \tag{6.1}$$

$$tpr\, \alpha = tg\, \alpha \tag{6.2}$$

where $0 \le k \le \infty$.

The index k define the "order" of the function. To $tpr\,\alpha$ function we did not attach any index k, since the value of this function does not depend on the value of k.

We denoted with $spr\,\alpha$, $cpr\,\alpha$ and $tpr\,\alpha$ the trigonometric functions similar to those of the Classical Trigonometry (CT) [21] but specific to each form of trigonometry mentioned above. As we have shown in the previous chapter, these forms are defined in function of the values domain of the order k. Thus:

- for IT, we have $0 \le k < 1$;
- for TT, we have $1 < k < 2$;
- for UT, we have $2 < k \le \infty$.

The borders between the above domains characterize, in their turn, the following trigonometries:

- for QT, we have $k = 1$;
- for CT, we have $k = 2$.

In the Polygonal Trigonometry (PT) (Chapter 2) k varies in the domain $1 < k < 2$, in function of the sides number of the "Basic polynomial" taken into consideration.

6.2. The Paratrigonometry, a general notion which includes CT, QT, PT, IT, TT and UT

In order to be able to look on the whole all these "partial" trigonometries individually valid for a certain value domain for k, we introduced this notion of "Paratrigonometry" (PRT) which is valid for the entire domain $0 < k \le \infty$.

The above mentioned trigonometries (IT, QT, TT, C, UT, PT) can be considered "components" of PRT. They are pointed out in Table 6.1, where we represent the entire structure of PRT.

The "categories" as part of PRT, mentioned in Table 6.1 are separated by the Quadratic Trigonometry (QT). This border component between IT and TT represents itself a "category", since there are some symmetries regarding basic trigonometric functions (BTFs) relate to QT, as we will se as follows.

Table 6.1

The Paratrigonometry (PRT) structure

The name of PRT structure	**The value domain of the order k**	**BTF form**	**The category as part of PRT**	**The generic denomination**
Infratrigonometry (IT)	$0 \leq k < 1$	Rhombus with curves sides and the concavity opposite to the reference 0*	Infratrigonometry (IT)	Paratrigonometry (PRT) $0 \leq k \leq \infty$
Quadratic Trigonometry (QT)	$k = 1$	Rhombus with straight sides (Rhombus with "Paratrigonometric mirrors")	Quadratic Trigonometry (QT)	
Transtrigonometry (TT)**	$1 < k < 2$	Rhombus with curves sides and the concavity towards the reference 0*	Extratrigonometry (ET) $1 < k \leq \infty$	
Classical Trigonometry (CT)	$k = 2$	Trigonometric circle		
Ultratrigonometry (UT)	$2 < k \leq \infty$	Rhombuses with curved sides and the concavity towards the reference 0*		

* The coordinate system Ox – Oy origine.

** Polygonal Trigonometry (PT) is referred to the domain $1 < k < 2$, but k is variable in function of the sides number in the Trigonometric Polygon.

Having this in consideration, the formulas (6.1) and (6.2) from above receive a specific expression to every component of PRT which we choose to refer to. Thus, for example, in TT they are expressed in the following way:

$$|st_k\alpha|^k + |ct_k\alpha|^k = 1 \tag{6.3}$$

$$tgt\,\alpha = tg\,\alpha \tag{6.4}$$

The index k was attached to the notation of the functions "sine", "cosine" etc., to define its order, and the introduction in formula (6.1) of the absolute values of the respective functions was imposed by the validity need of these formulas for odd and fractional values of k.

Evidently that for IT, in formula (6.1) we will have the notations $si_k\,\alpha$, $ci_k\,\alpha$, for UT the notations $su_k\,\alpha$, $cu_k\,\alpha$ etc.

In the light of what we have said before, in CT formula (6.1) is written under the known form:

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{6.5}$$

and in QT this has the form:

$$|sq\,\alpha| + |cq\,\alpha| = 1 \tag{6.6}$$

where $sq\,\alpha$ is "quadratic sine of angle α" and $cq\,\alpha$ is "quadratic cosine of angle α".

The formulas (6.2) and (6.4) respectively have a general character and thus we can write:

$$tpr_k\,\alpha = tgt_k\,\alpha = tgi_k\,\alpha = tgu_k\,\alpha = tq\,\alpha = tp\,\alpha = tg\,\alpha\ . \tag{6.7}$$

From above mentioned chapter regarding the trigonometric formulas analyzed, we see that the principal trigonometric functions of IT ($si_k\,\alpha$, $ci_k\,\alpha$) are characterized by the fact that their graphic representation have an "Arabian Archivolt". This thing is valid and for the principal trigonometric functions in TT case and thus also, for the border between IT and TT, namely for QT. the curves that represent these functions with respect to α are "cusp" points for $\alpha = \pi/2$, $\alpha = 3\pi/2$ etc. in the "sine" functions case, and respectively $\alpha = 0$, $\alpha = \pi$ etc., for "cosine" function.

The principal trigonometric functions in UT and CT inclusively, have $spr_k\alpha$ and respectively $cpr_k\alpha$ curves with a monotonous variations, without cusps. These functions make an exception for $k = \infty$ in UT. In this case these cusps appear for $\alpha = \pi/4$, $\alpha = 3\pi/4$ etc., if we refer to $spr_\infty\alpha$. Evidently, these cusps appear for same α values for the case of function $cpr_\infty\alpha$ also (see Chapter 5).

Therefore, from the point of view of the principal trigonometric functions graphical form, CT ($k = 2$) represents a border domain. Up to this ($0 \leq k < 2$) the curves $spr_\infty\alpha$ and $cpr_\infty\alpha$ have cusps and moreover they are inscribed in the interior of the curves $spr_\infty\alpha = \sin\alpha$ and $cpr_\infty\alpha = \cos\alpha$. Starting with $k = 2$, thus for $2 \leq k < \infty$, the respective curves do not have cusps. For the extreme case $k = \infty$, these cusps appear again. All these curves for the entire domain $2 < k < \infty$ are circumscribed to the characteristics curves for $k = 2$, thus the classical sinusoid.

Regarding the Basic Trigonometric Figures (BTFs) – Figure 6.1 – we mark the fact that in IT case, for $0 < k < 1$, these have a form of rhombuses with curved sides, and the concavity in the opposite direction to the reference point O of the coordinate axis Ox – Oy; for the limit situation $k = 0$, BTF consist on the cross OM – OP – ON – OQ.

For TT and UT cases ($1 < k \leq \infty$), BTFs form is of some rhombuses with curved sides and concavity to the direction of the reference point O, thus in an opposite sense to the concavity of the rhombuses with curved sides in IT. At the limit, when $k = \infty$, the curvature of these sides becomes extreme, they fragments in the points A, B, C and D. In Figure 6.1, in this way, BTF for $k = \infty$ becomes the square ABCD.

In consequence, it is to notice the fact that, from the BTFs point of view, the form of these figures can be part of two categories, namely: first, the one characteristic to IT and the second, characteristic to TT and UT. The limit between these two categories is QT ($k = 1$) where BTF is the rhombus with straight sides (without concavity) QMNP.

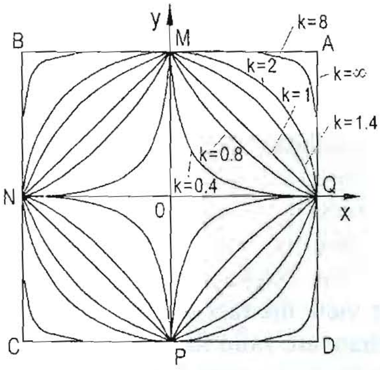

Fig. 6.1. The Basic Trigonometric Figures (BTFs) aspect in the entire Paratrigonometric (PT) domain ($0 \le k \le \infty$).

In the second category, a special case is the characteristic "trigonometric circle" in CT ($k = 2$).

Finally, we can say that, from the point of view of the graphical representation of $spr_k\alpha$ and $cpr_k\alpha$ functions, the border between these two distinct zones is the trigonometric circle, and from the point of view of BTFs, the border between the two distinct zones is the trigonometric rhombus with the straight line in QT. The zone between these two borders characterized by $1 < k < 2$, we named it as Transtrigonometry (TT) – see Chapter 3.

For our analysis which we carry on, because of some reasons which we will explain in detail as follows, we will have in our view as a guide the trigonometric rhombus with straight sides (BTF in QT). This divides the entire domain $0 \le k \le \infty$ in two areas, namely: IT ($0 \le k < 1$) and respectively TT ($1 < k < 2$) together with UT ($2 < k \le \infty$), including CT ($k = 2$). We named these last two trigonometries (TT and UT) together with the "Extratrigonometry" (ET) which, regarding to Figure 6.1, takes possession of the entire zone exterior to the rhombus QMNP ($k = 1$).

Regarding Table 6.1, we have to mention that the reasons for which we name the trigonometric rhombus in CT as the "Paratrigonometric mirror" will be explained here below. So we will analyse some noticed particularities of symmetries regarding BTFs of PRT.

6.3. The Basic Trigonometric Figures (BTFs) of Paratrigonometry and their implicit characteristic equations

In Figure 6.1, we represented the most important BTFs of PRT, namely: the Cross ($k = 0$), the Trigonometric Rhombus with straight sides MNPQ ($k = 1$), the Trigonometric Circle ($k = 2$) and the Trigonometric Square ABCD ($k = \infty$). Also, there are represented some of BTFs of IT ($k = 0.4$ and $k = 0.8$), TT ($k = 1.4$) and UT($k = 8$).

From their analysis we observe two symmetries, namely:

a) The curve symmetry in relation with the two coordinate axis for a specific value of k, which is found in one of the trigonometric quadrants;

b) The symmetry of a such curve situated "above" to the rhombus side QM (if we refer to the first quadrant) with a corresponding curve from the space "below" the respective side. In another way saying, BTFs from ET have symmetries in IT and conversely. The line QM is its own symmetry and all other symmetries are related to this one like being a mirror. For this reason we named BTF for QT (k = 1) "The Rhombus with paratrigonometric mirrors" (see Table 6.1), having in our view the fact that everything what we have said before regarding to the first quadrant are valid for the other (II – IV) quadrants, also.

In order to analyse from the mathematical point of view these symmetries, we apply to the formula which represent the side in the first quadrant – for simplicity – of BTF.

This formula was represented and explained in Chapter 3 and used in Chapter 4 and 5 and it is valid for the entire domain $0 \le k \le \infty$ and thus it represents a fundamental formula in PRT.

It express the relation between the coordinates x and y of the system Ox – Oy in which BTF is represented and has the form:

$$y = \left(1 - |x|^k\right)^{1/k}. \tag{6.8}$$

In Figure 6.2 there are represented the sides from the first quadrant of BTFs for $k = 1$; $1 < k < \infty$ (of ET domain) and $0 < \kappa < 1$ (of IT domain). We denoted by the Greek symbol κ the order value of the function from IT (to distinct from k of ET) for a reason which we will soon speak about it.

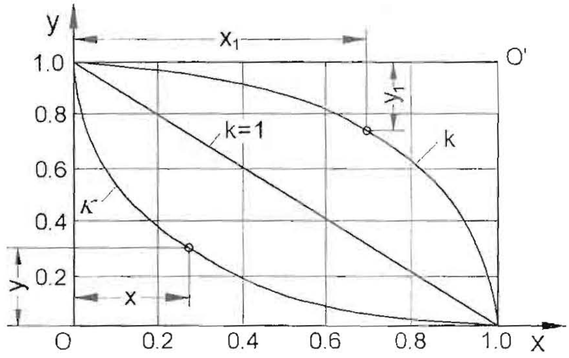

Fig. 6.2. Symmetric BTFs given the mirror k = 1, represented in the first quadrant.

In order to prove the symmetry of "a)" above it is sufficient to set formula (6.8) under the form:

$$x = \left(1 - |y|^k\right)^{1/k}. \tag{6.9}$$

The symmetry of formulas (6.8) and (6.9) direct us to conclude that there exists BTFs symmetry in relation with Ox and Oy coordinate axis.

Since we only refer to the first quadrant, as we have mentioned before, in formulas (6.8) and (6.9) it is no more in need to use "absolute values" for x and y (always positive in this quadrant) and the corresponding formulas can be written under a simpler form:

$$y=\left(1-x^{k}\right)^{1/k} \tag{6.10}$$

$$x=\left(1-y^{k}\right)^{1/k}. \tag{6.11}$$

In what follows, we will use formula (6.11) applied in ET, as we have mentioned before. On the other side, for IT this formula will have the form:

$$y=\left(1-x^{\kappa}\right)^{1/\kappa}. \tag{6.12}$$

To analyze the symmetry type mentioned at the point "b)" above, we consider that the curves characterized by k and κ of Figure 6.2 are symmetric with respect to the line $k = 1$. In another way said, the curve κ is the "mirror" image (with respect to $k=1$) of curve k, and conversely.

We denote the present coordinate of curve κ with x and y, and the present values of curve k with x_1 and y_1. We apply formulas (6.10) and (6.12) for the curves k and κ case. For curve k case we have:

$$y_1=\left(1-x_1{}^{k}\right)^{1/k} \tag{6.13}$$

and for curve κ case, formula (6.12) is also valid.

Referring to Figure 6.2, we can write $x_1=(1-x)$ and $y_1=(1-y)$. Introducing x_1 and y_1 in formula (6.13) in this way expressed as functions of x and y and using formula (6.12) to express y as function of x, we obtain the relation:

$$1-\left[1-(1-x)^{k}\right]^{1/k}=\left(1-x_1{}^{\kappa}\right)^{1/\kappa}. \tag{6.14}$$

If we take logarithm of relation (6.14) we obtain the following banding formula between κ and k:

$$\kappa=\left[\ln\left(1-x^{\kappa}\right)\right]\Big/A \tag{6.15}$$

where

$$A=\ln\left\{1-\left[1-(1-x)^{k}\right]^{1/k}\right\}. \tag{6.16}$$

We see that from formula (6.15) we can not find κ explicitly. In other words, formula (6.15) is implicitly given regarding κ (and also k).

In any case, the values of κ and k characterize two symmetric curves in ET and respectively in IT given the "mirror" having $k = 1$. If we symbolize the symmetry "status" by "Σim", we can write $\Sigma\mathrm{im}\ k \to \kappa$. Evidently, we can also write $\Sigma\mathrm{im}\ \kappa \to k$, or simply $\Sigma\mathrm{im}\ k \leftrightarrow \Sigma\mathrm{im}\,\kappa$.

Since κ and k respectively can be find in an implicit form in the formula (6.15) to determine κ as a function of k we proceed graphically as we will continue to show.

On the left side of the equality sign in formula (6.15) we replace κ by z. In this way we will obtain the following formula which express z as a function of κ:

$$z = \left[\ln\left(1 - x^{\kappa}\right)\right]/A \qquad (6.17)$$

where A is given by formula (6.16).

In this way $z = z(\kappa)$ have both parameters k and κ. Giving various values for k (in the domain $1 < k < \infty$) and for κ (in the domain $0 < \kappa < 1$) we obtain various curves representing $z = z(\kappa)$, as it can be seen in the Figure 6.3.

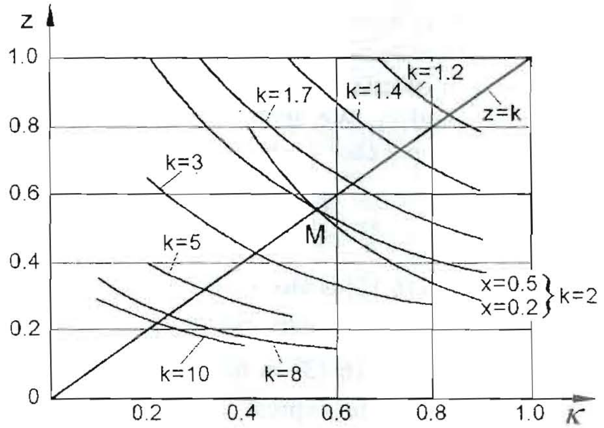

Fig. 6.3. The representation of the function $z(\kappa)$ for $x = 0.5$ (in the case $k = 2$), also for $x = 0.2$ having as parameter distinct values for k.

If for a certain value of k we live two values for x, the curves of the function $z(\kappa)$ intersect in a point whose coordinates represent the solution of the problem, namely $z = \kappa$. In Figure 6.3, M is a such point which represent the intersection of the curves $z(\kappa)$ for $k = 2$ and for the values of x = 0.5 and x = 0.2 respectively. The result is $\kappa = 0.56$. Thus we can write:

$$\Sigma\mathrm{im}\ (k = 2) \leftrightarrow \Sigma\mathrm{im}\ (\kappa = 0.56). \qquad (6.18)$$

Any other curve of the function $z(\kappa)$ for $k = 2$ corresponding other values of x will pass through the point M. This is because for any value of x the curve for κ of Figure 6.2 is symmetric with respect to the "mirror" $k = 1$, of the curve characterized by k. Evidently, the converse is valid also, as it is shown by formula (6.18).

In Figure 6.3 we see that the point M is on the line OM which represents the bisector of the right angle formed by the coordinate axis Ox – Oy. In fact the equation of this line is even $z = z(\kappa)$. The intersection of this line with other curves, for various values of k (see Figure 6.3), will give the solutions for $\Sigma\text{im } k \to \kappa$.

6.4. The semiempirical equation for the basic symmetric figures from Paratrigonometry

With values for $\kappa = \Sigma\text{im } k$, from Figure 6.3, we construct the curve $\kappa = \varphi(k)$ represented in Figure 6.4.

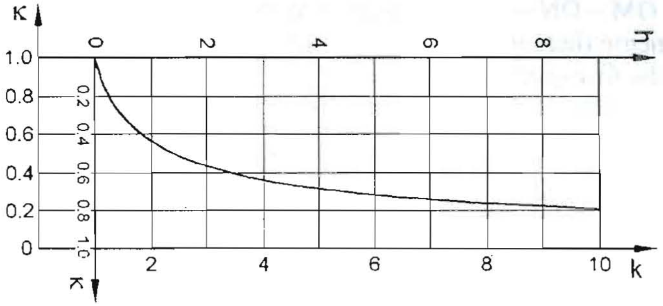

Fig. 6.4. The representation of function $\kappa = \varphi(k)$.

We see that, if considering and another system of coordinates than the system with axis Ok – Oκ, namely Ok' – Oκ', the curve mentioned before will have a form of the type representing exponential functions of the form:

$$k' = C \cdot e^{a \cdot \kappa'} \tag{6.19}$$

In order to have this function (6.19) expressed in the κ and k coordinates we replace κ' and k' by $\kappa' = (1 - \kappa)$ and $k' = (1 - k)$.

After making these substitutions, we apply again logarithm to the relation (6.19) and going the necessary operations for simplification we arrive to the following formula:

$$\kappa = \varepsilon - \left[\ln(k-1)\right]/a . \tag{6.20}$$

By trying some values for constants ε and a, and comparing the resulting values from formula (6.20) with those of Figure 6.4, we obtain $\varepsilon = 0.56$ and $a = 6$. With these values, formula (6.20) becomes:

$$\kappa = 0.56 - \left[\ln(k-1)\right]/6 . \tag{6.21}$$

This formula (6.21) represents, with a high precision degree, this function $\kappa = \varphi(k)$ for the values of k in the domain $1.075 < k < 10$.

Formula (6.21) is not applicable for $k = 1$, but returning to formula (6.17) we conclude that for $k = 1$ introduced in formula (6.15) we have $z = \kappa$ only if $\kappa = 1$. That is

$$\Sigma\text{im}\,(k = 1) \leftrightarrow \Sigma\text{im}\,(\kappa = 1)\,. \tag{6.22}$$

This last relation represents the mathematical expression of the "mirror" in the Paratrigonometry.

For the extreme cases marked by values $k = 0$ and respectively $k = \infty$ we apply the results of Chapters 4 and 5 mentioned before. Thus, in Chapter 4 we proved that, in IT, for $k = 0$ (now denoted $\kappa = 0$), the BTF is the Cross (OQ – OM – ON – OP, Figure 6.1), and in Chapter 5 we proved that, in UT, for $k = \infty$, the BTF is the square ABCD of Figure 6.1.

From the geometrical point of view it can be seen in Figure 6.1 that the Cross OQ – OM – ON – OP is symmetric to the square ABCD, in relation with the paratrigonometric rhombus of the mirrors. The same thing we can see in Figure 6.2, referring to the first quadrant. Using the above symbols, this thing can be expressed as:

$$\Sigma\text{im}\,(k = \infty) \leftrightarrow \Sigma\text{im}\,(\kappa = 0)\,. \tag{6.23}$$

Again from the analysis of formula (6.21) we can see that for $k = 2$ we obtain $\kappa = 0.56$ as we shown in the previous chapter and we have found it using relation (6.18).

6.5. Conclusions of Chapter 6

6.5.1. All the "Trigonometries" which we anlysed in the previous Chapters 1, 2, 3, 4 and 5 together with the Quadratic Trigonometry (QT) [1] and the Classical Trigonometry (CT) can be comprised in the notion of the Paratrigonometry (PRT) developed in Subchapter 6.2 here above.

The PRT structure and its relation with all the others trigonometries is given in Table 6.1. The basic relations from PRT, (6.1) and (6.2) can be applied in the case of all mentioned trigonometries, distinguishing themselves by the values for the "order" k.

This is also mentioned in Table 6.1, where we point out other classification elements, namely the Basic Trigonometric Figures (BTFs), as could be the Trigonometric circle in CT, Trigonometric rhombus with straight sides in QT etc.

6.5.2. Regarding BTFs, in PRT we established some BTFs symmetries, such these among BTFs from the Extratrigonometry (ET) with $1 < k \leq \infty$ and BTFs from the Infratrigonometry (IT) with $0 \leq k < 1$ in relation with the "Paratrigonometric mirror" of QT with $k = 1$ amply analyzed in Subchapter 6.3 and 6.4 here above.

Thus, in Subchapter 6.3 we established a semiempiric explicite equation for the respective symmetries. For a better expression of these symmetries, in IT case the order of the trigonometric functions was denoted by κ and the k notation was reserved for ET. We introduced the symbol "Σim", and the symmetry between a BTF from ET (of order k) and a corresponding BTF of than from ET of order κ from IT, was denoted by $\Sigma\text{im } k \leftrightarrow \Sigma\text{im } \kappa$, the sign $\leftrightarrow$ indicating the respective relation reciprocally.

7. REGARDING THE PARATRIGONOMETRIC EQUATION OF THE CIRCLE

7.1. Introduction

It is known that the fundamental relation of Classical Trigonometry (CT), which establish the connection between the values of the functions “sine” (sin) and “cosine” (cos) of the angle α is the following:

$$\sin^2 \alpha + \cos^2 \alpha = 1 . \tag{7.1}$$

If we take in consideration the trigonometric circle (with radius $R = 1$) and denote axis Ox – Oy, as in Figure 7.1, by similitude with relation (7.1) we have the following well known equation of the trigonometric circle, expressed in an algebraic form:

$$x^2 + y^2 = 1 . \tag{7.2}$$

On the other side, from Paratrigonometry (PRT) – Chapter 6 – we know that symmetric figure (with respect to the line AB of Figure 7.1) of a circle quarter AB (a), with the concavity towards the reference O is the circle quarter AB (b), with the concavity opposite to the reference O. Also, in Chapter 6 we showed that the circle quarter AB (b) can be expressed byt the following algebrical equation in the coordinate system Ox – Oy:

$$x^{0.56} + y^{0.56} = 1 . \tag{7.3}$$

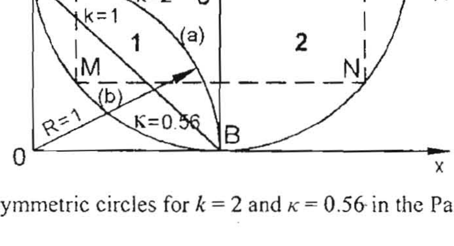

Fig. 7.1. The symmetric circles for $k = 2$ and $\kappa = 0.56$ in the Paratrigonometry.

Recall that the general fundamental equation in Paratrigonometry is:

$$x^{\kappa} + y^{\kappa} = 1 \tag{7.4}$$

for the "order" value κ in the domain $0 \le \kappa < 1$ (Infratrigonometry – IT). The corresponding equation for the order value (denoted with k in this case) in the domain $1 < k \le \infty$ (Extratrigonometry – ET) is:

$$y^{k} + x^{k} = 1\,. \tag{7.5}$$

Referring to equations (7.2) and (7.3) and using the established symbols in PRT, we repeat the following relation regarding the symmetry mentioned before:

$$\Sigma\mathrm{im}\ (k = 2) \leftrightarrow \Sigma\mathrm{im}\ (\kappa = 0.56)\,. \tag{7.6}$$

If we refer to Figure 7.1 and take into consideration the coordinate axis system O'X – O'Y we can write the classical equation of a complete circle which contains the arc AB (b), such that:

$$X^2 + Y^2 = 1\,. \tag{7.7}$$

We can not do the same thing if we desire to use only one equation of type (7.3) established in PRT and valid only for the quarter of the circle AB (b).

Here below we will establish the paratrigonometric equations of (7.3) type equation for the entire circle with radius $R = 1$.

7.2. Paratrigonometric equations for the entire contour of the circle with radius R = 1

Referring to Figure 7.1, we notice the following:

7.2.1. In the coordinate system Ox – Oy, the quarter of the circle AB (a) is algebraically expressed by equation (7.2) and its "symmetrical" (in the paratrigonometric terms) that is the circle quadrant AB (b) is algebraically expressed by equation (7.3).

7.2.2. If we change the origin of the coordinate axis from O (the coordinate axis being Ox – Oy) to O' (the coordinate axis being O'X – O'Y), the complete circle ABCD is algebraically expressed by equation (7.7).

On the other side, if we return to the coordinate axis Ox – Oy, the equation of the circle ABCD is the following:

$$(x-1)^2 + (y-1)^2 = 1\,. \tag{7.8}$$

As we have shown in the previous chapter the paratrigonometric equation (7.3) is valid for the circular arc AB (b) only and can not be extended to the entire circle using an equation of type (7.8).

Therefore, to establish the equations of type (7.3) to be also valid and for the rest of three quarters in the circle with center in O', respectively for the circle arcs

BC, CD and DA, we will use the symmetric property of the corresponding circle with respect to O'X and O'Y axis.

Thus, if on the circle arc AB (b) the point M of coordinates x and y satisfies equation (7.3), clearly on the circle arc BC the point N will satisfy the equation (7.3) too if we introduce the coordinates $(2 - x)$ and y for this point.

Thus, for the circle arc BC we have valid the equation:

$$(2-x)^{0.56} + y^{0.56} = 1. \tag{7.9}$$

For the circle arc CD:

$$(2-x)^{0.56} + (2-y)^{0.56} = 1. \tag{7.10}$$

For circle arc DA:

$$x^{0.56} + (2-y)^{0.56} = 1. \tag{7.11}$$

The equations (7.3), (7.9), (7.10) and (7.11) can be comprised in a single equation as such:

$$(A + B \cdot y)^{0.56} + (p + q \cdot x)^{0.56} = 1. \tag{7.12}$$

If we number all the four quadrants of the circle in Figure 7.1 successively, in a trigonometric sense, starting with the corresponding trigonometric quadrant (TQ) to the arc AB, then we will have the situation given in the table below (Table 7.1)

Table 7.1

The values of coefficients *A*, B, *p* and *q*, depending of *n*

TQ (*n*)	Circle arc	*A*	*B*	*p*	*q*
1	AB	0	1	0	1
2	BC	0	1	2	–1
3	CD	2	–1	2	–1
4	DA	2	–1	0	1

In this table we notice symmetry of the coefficients "p" and "q" with respect of the number "n" which represents the quadrant in reference. If we like to express algebraically this thing, this symmetry directs us to the idea that these respective coefficients can be second degree functions of variable "n". In this case, the graphical representation of these functions, $p(n)$ and respectively $q(n)$, could be that of Figure 7.2.

As it is known, the general equation for such function is, for example, in $p(n)$ case, as follows:

$$p(n) = C_1 \cdot n^2 + C_2 \cdot n + C_3 \tag{7.13}$$

where the coefficients C_1, C_2 and C_3 can be determined applying equation (7.13) for three points of known coordinates "p" and "n", as for example the points "0;1", "2;2" and "2;3".

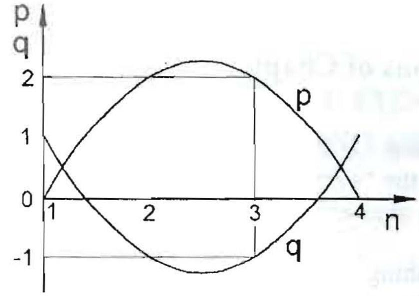

Fig. 7.2. The graphic representation of functions *p*(*n*) and *q*(*n*).

In this way, from equation (7.13) we obtain three equations with three unknowns C_1, C_2 and C_3. Because of function $p(n)$ symmetry and respectively of curve "*p*" from Figure 7.2, it was no need to use the "0;4" coordinate point. Solving this three equations mentioned above, we have $C_1 = -1$, $C_2 = 5$, $C_3 = -4$. Thus, equation (7.13) can be written as:

$$p = -n^2 + 5 \cdot n - 4 . \tag{7.14}$$

Similarly, we proceed also for function $q(n)$ and obtain the relation:

$$q = n^2 - 5 \cdot n + 5 . \tag{7.15}$$

For *A* and *B* coefficients this above reasoning is not valid and in consequence we applied the following arguments:

For *A* we can obtain value 0 (zero) in both cases when $n = 1$ and $n = 2$ if we have in view the product $[(n - 1) \cdot (n - 2)]$. Also with this product we can obtain the correct value for *A* when $n = 3$. That is $A(3) = 2$. The relation for *A* becomes applicable also for $n = 4$, if we write it under the form:

$$A = (n-1) \cdot (n-2) / |5 - 2 \cdot n| . \tag{7.16}$$

In our argument to establish the values of the coefficient B we consider the fact that this must change its algebraic sign from + (plus) to – (minus) when it passes from $n = 2$ to $n = 3$. This will happen with a difference of $(5 - 2 \cdot n)$ which also appears in the formula (7.16) above. In order to have values +1 and respectively −1 for the coefficient B we have to divide this difference with its absolute value. Thus we have:

$$B = (5 - 2 \cdot n) / |5 - 2 \cdot n| . \tag{7.17}$$

The circle of radius $R = 1$ in the coordinate system xOy (Figure 7.1) can thus be also represented by equations of degree distinct from two, namely of a degree smaller then 1 ($0.56 < 1$).

In this case we need four distinct equations, one for each circle quadrant individually. This thing was proved above, using specific paratrigonometric mathematical relations.

7.3. Conclusions of Chapter 7

7.3.1. The equation (7.3), fundamental in the Paratrigonometry (PRT) and for which it represents the "symmetry" of the circle represented by equation (7.2), but it is only valid for a quarter of the trigonometric circle of radius $R = 1$.

7.3.2. In establishing a relation of equation (7.3) type, having the exponent value equal with 0.56 for the component terms and with an extended validity in the entire trigonometric circle, we performed a reasoning based on the circle symmetry and thus we found equation (7.12). The coefficients which appear in this equation depend on the reference circle quadrant and can have values 0 (zero); +1; –1 and 2 respectively.

For each of the four circle arcs (corresponding each to one angle at the center equal with $\pi / 2$), which together form a complete circle, we indicate the values of these coefficients in Table 7.1, or they can be determined algebraically as a function of the order number (see Figure 7.1) of the referred quadrant (circle quadrant).

8. SOME DEVELOPMENTS OF FUNDAMENTAL PARATRIGONOMETRIC EQUATION

8.1. Introduction

It Chapter 3 we showed that the fundamental equation which represents the Basic Trigonometric Figure (BTF) in the Transtrigonometry (TT), if we refer only to the first quadrant, is:

$$y^k + x^k = 1 \tag{8.1}$$

where the value of k is the "order" of the transtrigonometric function of the case analyzed in the respective Chapter. In TT k has values in the domain $1 < k < 2$. From (8.1) results that BTFs in TT are rhombuses with curved sides, with the concavity oriented towards the reference O, which represents the coordinate axis Ox – Oy origin, where the equation (8.1) refers to.

Recall that in the Classical Trigonometry (CT) k =2; BTF in this case is the trigonometric circle having radius $R = 1$. In the Quadratic Trigonometry (QT) k = 1 but BTF in this case is the Trigonometric rhombus with straight sides.

For values of k in the domain $0 \leq k < 1$ we are in the domain of Infratrigonometry (IT) – see Chapter 4. BTFs in IT are also curved rhombuses as in TT case but the concavity of their sides is oriented in the opposite direction of O.

In the Paratrigonometry (PRT) – see Chapter 6 –, which comprise all of these above mentioned trigonometries, Ultratrigonometry (UT) inclusively having $2 < k \leq \infty$ (see Chapter 5) we evidenced some symmetries between BTFs in TT and UT (named together with CT inclusively as Extratrigonometry – ET) and IT. This symmetry is with respect to the straight of the rhombus representing BTF in Quadratic Trigonometry (QT).

In order to distinct BTFs of ET from BTFs of IT we denoted the "orders" of the respective functions by k, in ET case, and by κ, in IT case respectively.

In Figure 8.1 we represented the sides in the first trigonometric quadrant of a BTF from ET – the curve AB(k) – of its symmetry from IT – the curve AB(κ) – and of the trigonometric rhombus – the straight line AB(k = 1) characteristic to QT. Expressing the symmetry of the curves AB(k) and AB(κ) mathematically, we can write $x_1 = 1 - x$ and $y_1 = 1 - y$.

From the formula (8.1) we have

$$y = \left(1 - x^k\right)^{1/k}. \tag{8.2}$$

If we refer at the curve AB(k), the formula (8.2) become

$$y_1 = \left(1 - x_1^k\right)^{1/k}. \tag{8.3}$$

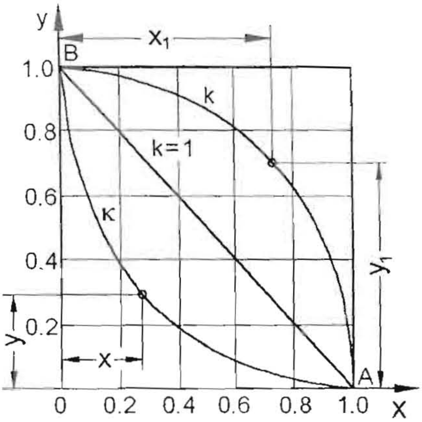

Fig. 8.1. BTFs, in first quadrant, for $k > 1$ (ET), $k = 1$ (QT) and $\kappa < 1$ (IT).

If we refer at the curve AB(κ), the formula (8.2) become

$$y=\left(1-x^{\kappa}\right)^{1/\kappa}. \tag{8.4}$$

Having in view the above mentioned relations between x_1 and x, and y_1 and y respectively, and the formulas (8.3) and (8.4) we arrive to the following basic relation for the symmetric functions in PRT (see Chapter 6):

$$\left(1-x^{\kappa}\right)^{1/\kappa}+\left[1-\left(1-x^{k}\right)\right]^{1/k}=1 \tag{8.5}$$

where

$$\kappa=0.56-[\ln(k-1)]/6. \tag{8.6}$$

One of the principal reasons of Chapter 6 was to determine formula (8.5).

We have to remember from this chapter that the relation (8.6) and implicity relation (8.5) are valid for the values of k in the domain $1.075<k\leq 10$.

Evidently, everything what we have discussed before is applied for under unit values of x and y (see Figure 8.1).

Here below we will consider the extension of the validity for equation (8.5) also for over unit values of the variable x.

8.2. Algebraic development of the fundamental equation for BTFs from PRT

If we would like that equation (8.2) to be valid for positive over unit values of x (let denote this case by X) we turn for help introducing a convenient X^{-1} in equation (8.5) and thus we have:

$$\left(1-X^{-\kappa}\right)^{1/\kappa}+\left[1-\left(1-X^{-1}\right)^{k}\right]^{1/k}=1. \tag{8.7}$$

In this way, we have to use the two equations, namely (8.5), for $0 \le x \le 1$ and (8.7) for $X > 1$.

These two equations can be combined in a single one namely:

$$\left(1-x^{a\cdot\kappa}\right)^{1/\kappa}+\left[1-\left(1-x^{a}\right)^{k}\right]^{1/k}=1 \tag{8.8}$$

where

$$a=(1-x)/\left|1-x\right|. \tag{8.9}$$

We see that for $0 \le x < 1$ result $a = 1$ and thus the equation (8.8) becomes the equation (8.2) and for $x > 1$ we have $a = -1$ and the equation (8.8) becomes the equation (8.7). If $x = 1$ by removing the indetermination in (8.9) we obtain $a = 1$ and we find again the first case when equation (8.8) becomes (8.5).

If we like to express by words relation (8.8) we can say that for any value of k in the domain $1.075 \le k \le 10$ and respecting relation (8.6), the relation (8.8) is valid for any positive rational value of x.

In other words, the equation (8.8) has an infinite number of roots that in all positive rational numbers.

Let denote by $f(x)$ the function of the left side of equality sign in equation (8.8). In this way, the equation (8.8) can be written as:

$$f(x)=1. \tag{8.10}$$

The graphical representation of this function in a coordinate system Ox – Oy, where $y = f(x)$, is the line $y = 1$.

8.3. The validity extension of the BTFs fundamental equation in PRT

The idea to extend the validity of equation (8.2) for values of x greater than the unit by introducing the coefficient "a" given by the relation (8.9) thus obtaining the equation (8.8), can be also useful in formula (8.1).

Thus, this becomes:

$$y^{k}+x^{a\cdot k}=1 \tag{8.11}$$

From this relation we obtain:

$$y=\left(1-x^{a\cdot k}\right)^{1/k}. \tag{8.12}$$

According to what we shown here above, formula (8.12) remain the same in the Extratrigonometry (ET) case, when $1 \le k \le \infty$. In the Infratrigonometry (1T) case formula (8.12) becomes:

$$y=\left(1-x^{a\cdot\kappa}\right)^{1/\kappa} \tag{8.13}$$

where κ is given as a function of k by formula (8.6). By the terms established in the PRT the curve representing the function y given by the formula (8.13), is the "symmetry" of the curve representing the function y given by formula (8.12), or using the symbol of Chapter 6, we can write:

$$\Sigma\text{im}\,(k) \leftrightarrow \Sigma\text{im}\,(\kappa) \tag{8.14}$$

Based on formula (8.14) and using the known result from PRT, evidently we can say that also the converse of what we have said before is valid, that is that the curve associated to the formula (8.12) is the "symmetry" of the curve associated to the formula (8.13).

Recall that this symmetry is formed with the respect of the line AB ($k = 1$) of Figure 8.1.

If we represent graphically function y of relation (8.12) for $x > 1$, we obtain the curves represented in Figure 8.2.

The curves representing function $y(x)$ for various values of κ and k respectively.

a. $\kappa = 0.236$ **e.** $k = 1.4$
b. $\kappa = 0.56$ **f.** $k = 2.0$
c. $\kappa = 0.713$ **g.** $k = 8.0$
d. $k = 1.0$

In order to have a complete image of this function graphical representation Figure 8.2 we also represented the respective curves for $0 \le k \le 1$, accepting k and κ as parameters with the limits $1.075 \le k \le 10$, as we had before.

In Figure 8.2 we limit our self to $x = 4$, because of space limit. In any case, for all values of k and κ, the curves for $k > 1$ are asymptotic with respect to the line $y = 1$. This also results from the relation (8.12) which for $x > 1$, thus $a < 0$ from relation (8.9) at the limit, when $x \to \infty$ it becomes:

$$\lim_{x\to\infty} y = 1\,. \tag{8.15}$$

We mention that in Figure 8.2 the represented curves are connected by the following symmetric relations: Σim (k = 8) ↔ Σim (κ = 0.713); Σim (k = 2) ↔ Σim (κ = 0.56) and Σim (k = 1.4) ↔ Σim (κ = 0.236).

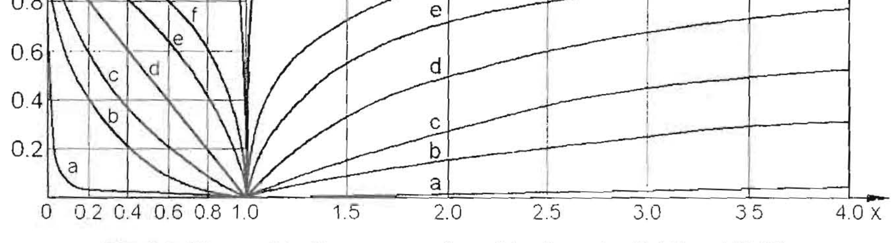

Fig. 8.1. The graphically representation of the formulas (8.12) and (8.13).

It is good to notice the fact that the curves of the function y go faster to the limit $y = 1$, when the values of k and κ respectively are large. Thus, we can see that for $k = 8$, the curve $y(x)$ is very close with the asymptote $y = 1$ starting with $x = 0.5$ and for $k = 2$ this closeness becomes more evident for $x = 4$.

On the other side, in the domain $\kappa < 1$ (thus in IT), for a very small value of κ (for example $\kappa = 0.236$) the curve $y(x)$ will move away very slowly (almost unnoticed for $x < 4$) from Ox axis.

Also, it is good to mention that all curves for $y(x)$ function in the domain characterized by $x > 1$ have their concavity oriented toward Ox axis, while the curves of this function in the domain $0 \leq x \leq 1$ have the concavity towards the Ox axis respective to the origin O, for $x > 1$ case and in a reverse direction, for $\kappa < 1$ case.

Since, for $x > 1$ the function $y(x)$ curves are developing toward infinity, in one way we can say that they represent the "projection toward infinity" of the curves for the respective function for $0 \leq x \leq 1$.

8.4. Conclusions of Chapter 8

8.4.1. The basic relation of the Paratrigonometry (PRT) – see Chapter 6 – elaborated on the bases of the principle of the Basic Trigonometric Figures (BTFs) symmetry, represents, in fact, an equation with an infinite number of solutions. In another way said, it is valid for any positive rational value of the variable x.

This means that the corresponding relation, enough complex do, having x as variable, is equivalent with a simple equality: $y = 1$.

8.4.2. In PRT the functions, which form the algebraic expression of BTFs were having until now the values in the domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Using a convenient calculation these functions can become valid also for the domain $1 \leq x \leq \infty$ and $0 \leq y \leq 1$. The representing curves for this functions when $x \geq 1$ start from the values $y = 0$ (for $x = 1$) and progress asymptotically towards $y = 1$ (for $x \to \infty$).

9. THE APPLICATION OF THE PARATRIGONOMETRIC FUNDAMENTAL EQUATIONS IN THE FOUR TRIGONOMETRIC QUADRANTS

9.1. Introduction

It is known from the Chapter 3 and 6 that the fundamental Basic Trigonometric Figures (BTFs) of the Paratrigonometry (PRT), written in the simplest form is:

$$y^k + x^k = 1 \tag{9.1}$$

where y and x are BTF coordinates and k is the "order" of the paratrigonometric function. The equation (9.1) is valid only in the first trigonometric quadrant (TQ), region in which both x and y have positive values.

Recall, as particular cases, the values $k = 2$ (Classical Trigonometry – CT) and $k = 1$ (Quadratic Trigonometry – QT). We see that k = 2 (CT) case the equation (9.1) is valid for all the four TQ, since the even power applied to the variables x and y cancel their negative values. Thus, the something is valid for all even number values of k, that is $k = 2 \cdot n$, when n is a positive natural number.

In Figure 9.1 we represent BTFs for $k = 1$ (in QT), $k = 2.5$ and $\kappa = 0.4924$ in the coordinate system xOy and we use the Roman notation (I – IV) for the four TQs. The value $\kappa = 0.4924$ represents the "symmetry" for value k = 2.5 in the signification of the characteristic symmetries in the PRT [6].

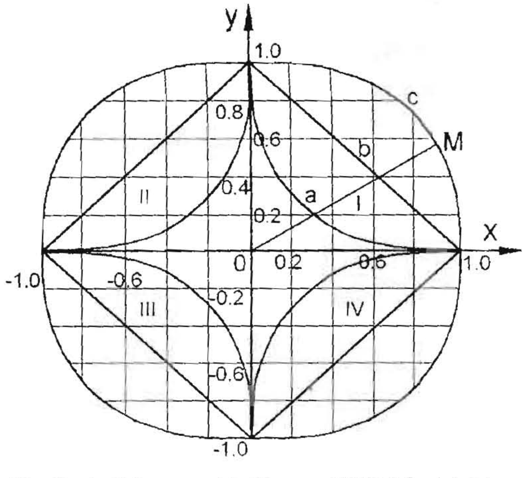

Fig. 9.1. The Basic Trigonometric Figures (BTFs) for $|x| < 1$ and $|y| < 1$: a – $\kappa = 0.4924$; b – $k = 1.0$; c – $k = 2.5$.

We mention that in PRT (see Chapter 6), in order that equation (9.1) to be valid in all the four TQs, was written under the form:

$$|y|^k + |x|^k = 1. \tag{9.2}$$

The equation (9.2) is satisfied for both positive and negative values of the variable x and y.

The problem of the algebraic signs + (plus) and – (minus) is coming when in (9.2) we solve for y in function of x. Thus we have

$$y = \pm\left(1 - |x|^k\right)^{1/k}. \tag{9.3}$$

In this case we must "choose" the sign + (plus) when we refer to TQs I and II and respectively the sign – (minus) when we consider the TQs III and IV. This choice is not mathematically objective, and when we operate with the relation (9.3) we have to use either the real image (geometrical) or the thought image of a BTF.

9.2. The algebraic signs coefficients + (plus) or – (minus)

As we known, the fundamental algebraic equation of the BTFs, written under the form (9.1) or (9.2) has a trigonometric correspondent, the fundamental trigonometric equations.

This establishes the relation between the basic function of type "sinus" and "cosinus" respectively. It is similar with equation (9.1) and (9.2) respectively, the variable y and x being replaced with the functions "sinus" and "cosinus" of the angle α, represented in Figure 9.1.

In PRT these functions are denoted by $spr\,\alpha$ and $cpr\,\alpha$ respectively (see Chapter 6). If in Figure 9.1 we refer to the corresponding BTF for $k = 2.5$, we will have $MN = spr\,\alpha$ and $ON = cpr\,\alpha$.

As we have shown, the equivalent equation to (9.1) is:

$$spr_k{}^k\,\alpha + cpr_k{}^k\,\alpha = 1 \tag{9.4}$$

where $spr_k\,\alpha$ is the "paratrigonometric sine of order k for angle α" and $cpr_k\,\alpha$ is the "paratrigonometric cosine of order k for angle α".

The equivalent equation with (9.2) is the following:

$$|spr_k\,\alpha|^k + |cpr_k\,\alpha|^k = 1. \tag{9.5}$$

If we solve for $spr_k\,\alpha$ in equation (9.5) we obtain the following relation equivalent to (9.3):

$$spr_k\,\alpha = \pm\left(1 - |cpr_k\,\alpha|^k\right)^{1/k}. \tag{9.6}$$

A similar relation we obtain when we solve for $cpr_k\, \alpha$ in (9.5).

Regarding to the choice of sign + (plus) or – (minus) in (9.6), it is valid in the same discussions performed regarding relation (9.3).

In the relation (9.6) case we can avoid the subjectivity referred when we analyzed relation (9.3) by introducing the following "algebraic sign coefficient":

$$b = (\pi - \alpha) / |\pi - \alpha| \tag{9.7}$$

where the angle α is expressed in radians, thus α [rad]. If angle α is expressed in degrees, the relation (9.7) becomes:

$$b = (180 - \alpha) / |180 - \alpha| . \tag{9.8}$$

Thus relation (9.6) becomes:

$$spr_k\, \alpha = b \cdot \left(1 - |cpr_k\, \alpha|^k\right)^{1/k} . \tag{9.9}$$

By the coefficient b the sign + (plus) or – (minus) in formula (9.6) automatically appears in formula (9.9). That is, if $\alpha < \pi$ (TQs I and II) in relation (9.7) then we have $b = +1$, and when $\pi < \alpha < 2 \cdot \pi$ (TQs III and IV) we have $b = -1$. When we consider relation (9.8), the same reasoning is valid, considering 180° instead of π (in radians) and expressing α in degrees, as we have shown before.

If we consider relations (9.7) and (9.8) for case $\alpha = \pi$, respectively $\alpha = 180^\circ$, the we have an indeterminate situation. In order to eliminate this indetermination we use the method to calculate the limits "superior" and "inferior" respectively, explained in Chapter 4. In this case we have:

$$\lim_{\alpha \to \pi^-} b = +1 \tag{9.10}$$

$$\lim_{\alpha \to \pi^+} b = -1 . \tag{9.11}$$

We see that even in the place $\alpha = \pi$ the algebraic sign changes from + (plus) to – (minus) in formulas (9.7) and (9.8). In $\alpha = 2 \cdot \pi$ case, the same reasoning is valid for the inferior limit. Thus:

$$\lim_{\alpha \to 2 \cdot \pi^-} b = -1 . \tag{9.12}$$

For the situation $\alpha > 2 \cdot \pi$ we have to apply to the "periodic" character of the trigonometric and paratrigonometric functions. In this sense, we reconsider the cycle valid for the interval $0 < \alpha < 2 \cdot \pi$. Thus, we can write:

$$\lim_{\alpha \to 2 \cdot \pi^+} b = +1 . \tag{9.13}$$

In this way, the angle α varying in a trigonometric sense, when it passes through the value $\alpha = 2 \cdot \pi$, the b coefficient passes from the value $b = -1$ to $b = +1$.

We mention that, for the algebraic sign coefficient above mentioned, we used the letter "*b*" since the letter "*a*" was used in Chapter 8 before, for another algebraic sign.

The problem of the algebraic sign for the function $y(x)$ expressed by the relation (9.3) which represents the mathematical expression of the BTFs can not be resolved in the same way as above. That is because in the corresponding relation the angle α does not interfere as it happens in the trigonometric functions case – se formula (9.6) – in order for the algebraic signs + or – to automatically appear in formula (9.3) in function of trigonometric quadrant (TQ) to which we refer, its is evident that we have to find a coefficient (symbolized by "c") that depends by the order number "n" of the reference TQ.

We express *c* by a similar relation with relations (9.8) and (9.9) respectively. Thus:

$$c = (m-n)/|m-n|. \tag{9.14}$$

For *c* to have the value $c = +1$ in $n = 1$ and $n = 2$ cases (TQs I and II) we must have $m > 2$. Similarly, for $c = -1$ in $n = 3$ and $n = 4$ cases (TQs III and IV) we must have $m < 3$. Both above conditions are thus satisfied when $2 < m < 3$. If $m = 2.5$, in this case formula (9.14) becomes:

$$c = (2.5-n)/|2.5-n| \tag{9.15}$$

and the relation (9.3) becomes:

$$y = c \cdot \left(1-|x|^k\right)^{1/k} \tag{9.16}$$

where *c* is given by (9.15), *n* being the order number of TQ, from 1 (TQ I) to 4 (TQ IV).

With a similar problem to establish some coefficients containing also the algebraic signs + or –, we have meet in Chapter 7. Summing up, of everything what we have mentioned above regarding the algebraic sign coefficients introduced in Paratrigonometry (PRT) in this Chapter and Chapter 8 we remember the following:

The coefficient *a* is given by the formula

$$a = (1-x)/|1-x|. \tag{9.17}$$

It was introduced in Chapter 8 in order to have only one relation linking the functions representing the BTFs and their "symmetries", as well for unit values of *x*, as for their upper-unit values.

The coefficient *b* was given by formulas (9.7) and (9.8) above and is used to establish "automatic" mathematically, the algebraic sign + or – of the $spr_k\,\alpha$ as a function of the values of the angle α – see relation (9.9).

The coefficient c given by relation (9.15) above is used to establish the algebraic sign of the function y which represents BTF corresponding to the paratrigonometric functions – see relation (9.16).

Figure 9.2 is an application example of the fundamental equation for BTFs, in the general case $-\infty < x < \infty$ (see Chapter 8), using the algebraic signs coefficients established in this Chapter.

In this Figure 9.2, we represented in all four TQ, the BTFs for $k = 2$, $k = 1$ and $\kappa = 0.56$. This last one ($\kappa = 0.56$) represents the "symmetry", in the paratrigonometric meaning, of a BTF characterized by $k = 2$. In Chapter 8 there is a similar figure, but only regarding to TQ I.

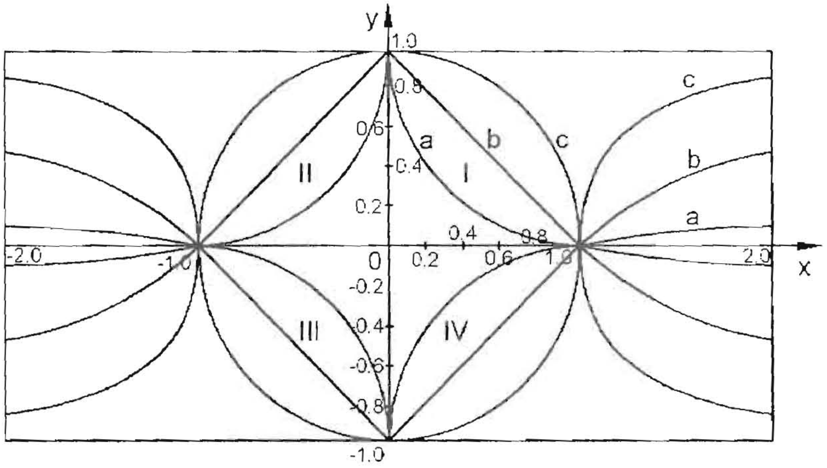

Fig. 9.2. BTFs for $0 < x < \infty$ and $|y| < 1$:
a – $\kappa = 0.56$; b – $k = 1.0$; c– $k = 2.0$.

9.3. Conclusions of Chapter 9

Currently, we attribute to the trigonometric functions the algebraic signs + (plus) or – (minus) in an intuitive way according to the Trigonometric Quadrant (TQ) – I to IV – which we are considering. Thus, as we well know, the function $\sin\alpha$ has positive values in the TQs I and II and negative values in the TQs III and IV. The function $\cos\alpha$ has positive values in the TQs I and IV and respectively negative values in the TQs II and III. The other trigonometric functions, being expressed in function of $\sin\alpha$ and $\cos\alpha$, will have the resulting algebraic signs.

Also, the algebraic signs of the coordinates x and y of the points on the curves which form the Basic Trigonometric Figures (BTFs), that is trigonometric circle in the Classical Trigonometry (CT) case are attributed in function of the reference TQ. Thus, the abscissa x has positive values in the TQs I and IV and negative values in the TQs II and III. The ordinate y has positive values in the TQs I and II and negative values in the TQs III and IV.

For all the above cases and with validity extended to the entire domain of the Paratrigonometry (PRT) in order to result mathematically (not intuitively) these mentioned above algebraic signs, we established the algebraic signs coefficients "*b*" and "*c*". The coefficient *b* is applied in the paratrigonometric functions case $spr\,\alpha$ and $cpr\,\alpha$, respectively. It is expressed as a function of angle α. The coefficient *c* is applied in the coordinates *x* and *y* case which define the referred BTF. It is expressed in function of the order number "*n*" of TQ in discussion. Evidently, for TQ I we have $n = 1$ and so on.

In this way, all the mathematical relations derived in the previous Chapters 3 and 8, which for simplification were established for TQ I, can be applied in all the others TQs (II – IV), where the algebraic signs + (plus) or – (minus) can be established mathematically, by computing the coefficients *b* and *c*.

10. AN APPLICATION CASE OF THE PARATRIGONOMETRIC POLAR COORDINATES

10.1. Introduction

In the Chapter 6 we presented the basic elements of the Paratrigonometry (PRT). We remember that the fundamental equations of PRT are the following:

$$|spr_k\alpha|^k + |cpr_k\alpha|^k = 1 \tag{10.1}$$

$$tpr_k\,\alpha = tg\,\alpha \tag{10.2}$$

where $spr_k\,\alpha$ is "the paratrigonometric sinus of order k of the angle α", $cpr_k\,\alpha$ is "the paratrigonometric cosinus of order k of the angle α" and $tpr_k\,\alpha$ is "the paratrigonometric tangent of order k of the angle α".

Relation (10.2) represents the connection key between the Paratrigonometry (PRT) and the Classical Trigonometry (CT). As a matter of fact, this well known CT represents a particular case of the PRT, which is characterized by $k = 2$. Accepting PRT, the trigonometric functions in CT, $\sin\alpha$, $\cos\alpha$, $tg\,\alpha$ etc., represent the paratrigonometric functions of order 2 of the angle α. Another particular case of PRT is the Quadratic Trigonometry (QT) where $k = 1$. The bases of QT were given by Valeriu Alaci, professor of the University "Politehnica" of Timisoara from Romania [1].

From the relations (10.1) and (10.2) we can calculate the functions $spr_k\,\alpha$ and $cpr_k\,\alpha$, for any value of the "order" ($0 \le k \le \infty$), as a function of $tg\,\alpha$. Thus, for example:

$$spr_k\alpha = \pm|tg\,\alpha| \Big/ \left(1 + |tg\,\alpha|^k\right)^{1/k}. \tag{10.3}$$

The sign + (plus) or – (minus) is given to the function $spr_k\,\alpha$ by the known rule in CT, that is in function of the trigonometric quadrant where the angle α is situated.

The paratrigonometric functions are analyzed with the reference to the Cartesian coordinate system. In this system, the most frequently used in Mathematics, in PRT case the variable – angle α – is horizontally represented on the abscissa axis and the functions $spr_k\,\alpha$, $cpr_k\,\alpha$ etc. are represented vertically on the ordinate axis.

In addition to the paratrigonometric functions an important roles in PRT have also the "Basic Trigonometric Figures" (BTFs). Generally, these represent what the "trigonometric circle" (with radius $R = 1$) is in CT and respectively the "trigonometric rhombus" (where all angles are right angles) in QT.

In the Chapter 6 we determined that the BTF equation is:

$$|y|^k + |x|^k = 1 \tag{10.4}$$

where k is the "order" of the paratrigonometric function to which is the respective BTF referring – see relation (10.1) and (10.2). Thus, as an example, for the trigonometric circle, characteristic for CT, $k = 2$ and for the trigonometric rhombus, characteristic for QT, $k = 1$.

As in a normal way, the paratrigonometric functions are represented in the Cartesian coordinate system, also BTFs are represented in these coordinates.

In Figure 10.1 we represent, in the Cartesian coordinate system with the abscissa Ox and ordinate Oy, all the BTFs for $k = 1$, $k = 2$, $k = 4$ and $k = \infty$ in all four quadrants (I – IV).

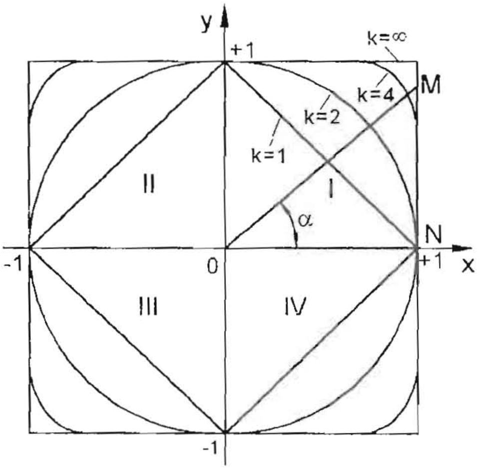

Fig. 10.1. Basic Trigonometric Figures (BTFs) applied in Paratrigonometry (PRT) for $k = 1$, $k = 2$, $k = 4$ and $k = \infty$.

It is interesting to remark the closeness between our results in regard to the BTF, totally independent obtained in comparison with the "quadrilobes", ample analyzed with a very developed and sophisticated mathematical tool by the professor Mircea Eugen Şelariu of University "Politehnica" of Timisoara, Romania [21].

We consider that the Paratrigonometry can have multiple technical applications in engineering. On the other way, in the technique we see many processes where rotation motion intervenes and these, in there turn, can be very well mathematically modeled using these polar coordinates.

Here below we will analyze one of these cases where the polar coordinates in PRT are applied.

10.2. The paratrigonometric functions represented in polar coordinates, applied in the technology of toothed wheels

The toothed wheels are very often used in the domain of the machines and mechanical installations in the diverse transmission systems with chain, conveyers, elevators, the rolling systems with caterpillars and especially gearings [12]. In function of their utilization the teethes profile have diverse geometrical forms; this problem does not constitute the subject of this Chapter.

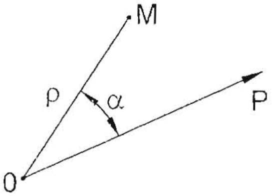

Fig. 10.2. Defining elements of a polar coordinate system.

The characteristic elements of a system in polar coordinates (see Figure 10.2) [15] are the following:

– the pole O which constitutes the origin point for polar axis Op and for polar radius ρ;

– the polar axis Op in function of which the polar angle value α is measured;

– the polar radius ρ measured from the origin O to the point M whose position is determined by the polar coordinates α and ρ;

– the polar angle α.

The position of the point M for which a specific value of the polar angle α is well determined if the polar radius ρ given by the function $\rho = \rho(\alpha)$ is known. The function $\rho =$ ct. represent the mathematical model of a circle with the center in O having the radius $R = \rho$. Thus, for example, the trigonometric circle in QT is represented by the function $\rho = 1$.

Returning to the toothed wheels we remember that actually the designing and their machining is performed "tooth by tooth". The periphery of the toothed wheel is performed by placing on it of an integer number of the successive pairs tooth-gap.

In what is following we propose to establish the mathematical model for the entire denture of toothed wheel. If we use unit measurements for the angles this denture is developing on that angle which characterize a whole circle, namely $(2 \cdot \pi)$ rad. or 360°. It is critical that, in this given situation, is convenient or almost mandatory to use polar coordinates. Since we will work with very small angles we choose as measurement units the "degree" and not the "radian". We consider that the number of the teeth composing the denture of the toothed wheel is z. They are disposed on the circle with the average polar radius ρ_m, as it is represented in Figure 10.3. The total height of the tooth is $(2 \cdot h)$. The head of the tooth with the height h is its portion between the average radius ρ_m and exterior radius ρ_e. The foot of the tooth, having the same height h, is its position between the interior radius ρ_i and ρ_m. The polar radiuses mentioned as well as the current radius ρ, they all have the origin in the pole O, which coincide with the center of the toothed wheel. For a total number of teeth (in fact pairs tooth-gap) z means that the angle which corresponds to a such pair is $\alpha_z = (360/z)^{\circ}$. If we desire to express mathematically the contour of such pair by the periodic function $spr_k\,\varphi$, for example, means that it has to reproduce a complete variation for a period of 360° in the course of the angle α_z only. This condition is fulfilled when the variable φ of the function $spr_k\,\varphi$ given by the product $\varphi = z \cdot \alpha$, where φ is the polar angle. In this way, with α_z we have $\varphi = 360^{\circ}$. Thus the complete period for the $spr_k\,\varphi = spr_k\,(z \cdot \alpha)$ is α_z. During of this period the function $spr_k\,(z \cdot \alpha)$ varies having successive values 0 and respectively +1 and –1.

The order of the function, represented by k, is that one which is dictated in our case by the tooth profile form. For simplification, we accept $h = 1$. In this case, considering what we have shown above, the polar radius ρ for the points which form the denture profile of a toothed wheel is given by the relation:

$$\rho = \rho_m + spr_k\,(z \cdot \alpha). \tag{10.5}$$

Based on this relation, in the Figures 10.3 ... 10.6 we represented the contours of three teeth, for $\rho_m = 6 \cdot h$ and for the four values of k for which we represented BTFs in Figure 10.1.

Thus, in Figure 10.3 we represented the case $k = 1$, in Figure 10.4 the case $k = 2$, in Figure 10.5 the case $k = 4$ and in Figure 10.6 we refer to the case $k = \infty$.

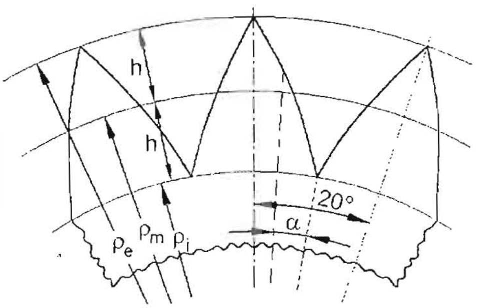

Fig. 10.3. Segment (with three teeth) for a paratrigonometrically defined denture for $k = 1$.

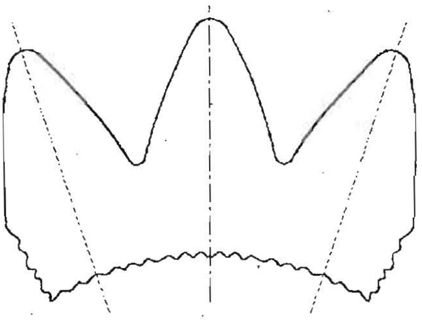

Fig. 10.4. Segment (with three teeth) for a paratrigonometrically defined denture for $k = 2$.

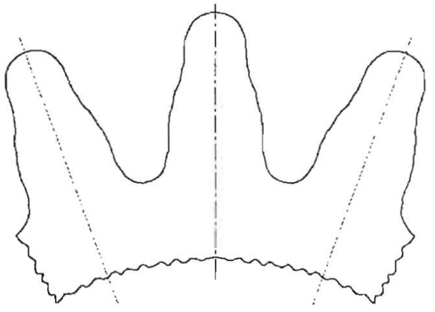

Fig. 10.5. Segment (with three teeth) for a paratrigonometrically defined denture for $k = 4$.

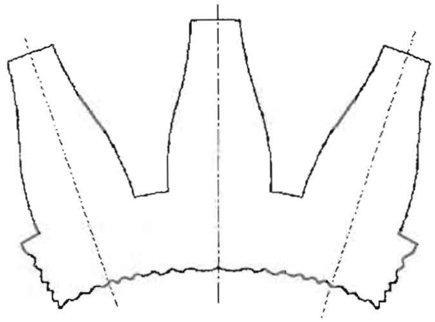

Fig. 10.6. Segment (with three teeth) for a paratrigonometrically defined denture for $k = \infty$.

Regarding the situation when $k = \infty$ (Fig. 10.6) we like to mention that under an algebraic aspect it is about to solve some problems in a "limit" case as we proceeded in a previous Chapter 5. For a better understanding of the function $spr_{\infty}\,\alpha$ variation we can appeal to the BTF representation for $k = \infty$ of Figure 10.1. We see that in the first trigonometric quadrant, in the domain $0^{\circ} \leq \alpha \leq 45^{\circ}$, $spr_{\infty}\,\alpha$ is represented by the line segment *MN*. On the other side $tg\,\alpha = MN\,/\,ON$ and since $ON = 1$, we will have $spr_{\infty}\,\alpha = tg\,\alpha$. For the domain $45^{\circ} \leq \alpha \leq 90^{\circ}$ we will have $spr_{\infty}\,\alpha = 1$, as we can see in Figure 10.1.

For the other trigonometric quadrants (II ... IV) we will have $|spr_{\infty}\,\alpha| = |tg\,\alpha|$, for the domains $135^{\circ} \leq \alpha \leq 180^{\circ}$; $180^{\circ} \leq \alpha \leq 225^{\circ}$ and $315^{\circ} \leq \alpha \leq 360^{\circ}$. For the other domains of the angle α, we will have $spr_{\infty}\,\alpha = \pm 1$, as we can see in Figure 10.1. The signs + (plus) or – (minus) should be applied, in all cases, according the known rule from CT.

Of course, for our case represented in Figure 10.6, everything what we have shown above regarding the domains of the values for α are valid regarding the angle $\alpha = (z \cdot \alpha)$.

10.3. Conclusions of Chapter 10

Considering what we have analyzed in the previous chapters we have the following important conclusions:

10.3.1. In using the Paratrigonometry for applications there are situations when it is recommended to use the polar coordinates instead of the Cartesian coordinates.

10.3.2. A situation for which the use of the polar coordinates is preferable is the paratrigonometric modeling of the toothed wheels dentures.

This modeling permit a unique relation to represent of the entire denture of the toothed wheels, in contrast with the present situation when their designing and machining is performed using the method "tooth with tooth". The relation which represents the entire denture of the toothed wheel contains the average polar radius of this denture and the paratrigonometric function "paratrigonometric sinus of order *k*" of the product between the current polar angle and the total number of the teeth.

10.3.3. The teeth profiles is a function of the value of the paratrigonometric order *k*.

11. PARATRIGONOMETRIC FUNCTIONS RAISED TO SOME POWERS AND THEIR APPLICATIONS

11.1. Introduction

In Chapter 6 the authors showed that the basic functions "Paratrigonometric sinus of α", denoted $spr_k\,\alpha$, has various values and consequently various graphical forms depending of the "order" value k. In Figure 11.1 is represented the function $spr_k\,\alpha$ for various k values such as: $k = 0.25$, $k = 0.5$, $k = 1.0$, $k = 2.0$, $k = 4.0$ and $k = \infty$.

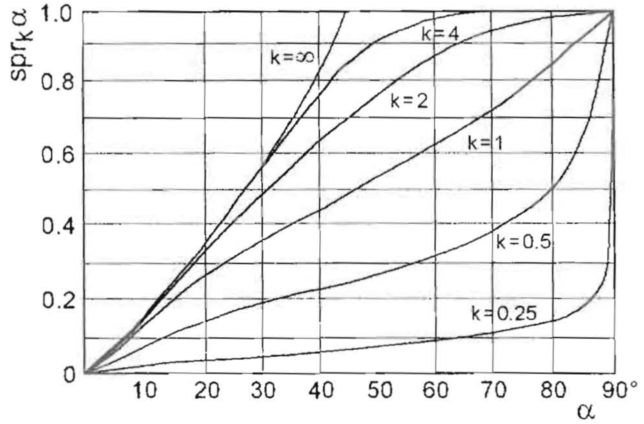

Fig. 11.1. The paratrigonometric function $spr_k\,\alpha$ for 6 values of the "order" k.

We recall that the fundamental relation in the Paratrigonometry is the following:

$$|spr_k\alpha|^k + |cpr_k\alpha|^k = 1 . \tag{11.1}$$

Thus when $k = 2$, the relation (11.1) becomes the very well known relation of the Classical Trigonometry (CT) as such:

$$(\sin\alpha)^2 + (\cos\alpha)^2 = 1 . \tag{11.2}$$

The relations (11.1) and (11.2) indicate the fact that CT represents a particular case of the Paratrigonometry (PRT), when $k = 2$.

The "key" connection between PRT and CT is the following relation:

$$tpr_k\,\alpha = tg\,\alpha\,. \tag{11.3}$$

This relation is valid for any value of k.

From the relations (11.1) and (11.3) we can obtain the following relation for $spr_k\,\alpha$ which can be calculated from the function $tg\,\alpha$, very well known from the CT:

$$spr_k\,\alpha = tg\,\alpha \,/ \left[1 + (tg\,\alpha)^k\right]^{1/k}. \tag{11.4}$$

Evidently that all the other paratrigonometric functions (cosine paratrigonometric etc.) can be calculated using the relations (11.1) and (11.4).

Returning to the Figure 11.1, we can see that in the spaces between the traced curves we can insert an infinite number of curves developed between the points with the coordinates (0°; 0) and (90°; 1.0), corresponding to the other values of k in the domain $0 \le k \le \infty$.

All these curves "fill in" the space limited by $O\alpha$ axis, the vertical line $\alpha = 90^\circ$ and the curve which represent the function $spr_\infty\,\alpha$. They all lock as a "bunch of fibers" which start from a single point and terminate in the other, but each fiber has the form dictated by the corresponding $spr_k\,\alpha$ function respectively. We mention that the segment $O\alpha$, between the lines $\alpha = 0^\circ$ and $\alpha = 90^\circ$, together with the segment on the vertical line $\alpha = 90^\circ$, between the limits $spr_k\,\alpha = 0$ and $spr_k\,\alpha = 1.0$, represents $spr_k\,\alpha$ for $k = 0$, as we have shown in Chapter 4.

In order to represent some periodic functions with a sinusoidal like form, which we meet very often in Physics and Technology respectively, we consider that there is a need for other "traces" of the above mentioned "fibers". For this purpose we can apply to the function $spr_k\,\alpha$ raised to some powers, thing that raise very much the area of the paratrigonometric modeling of some periodic functions with sinusoidal forms. If we denote by p the power at which is raised the paratrigonometric function $spr_k\,\alpha$ we can write $(spr_k\,\alpha)^p$.

In Figure 11.2, as an example, we traced the curves which represent the function $(spr_k\,\alpha)^2$ and in Figure 3 we traced the curves which represent the function $(spr_k\,\alpha)^{0.5}$ for the same values of order k, as in Figure 11.1.

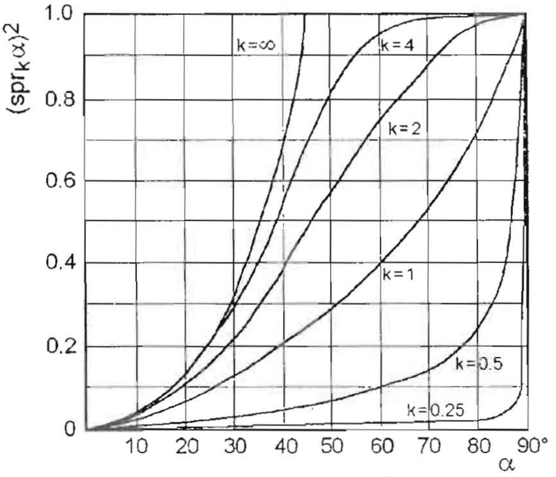

Fig. 11.2. The function $(\mathrm{spr}_k\alpha)^2$ for the k values according the Figure 11.1.

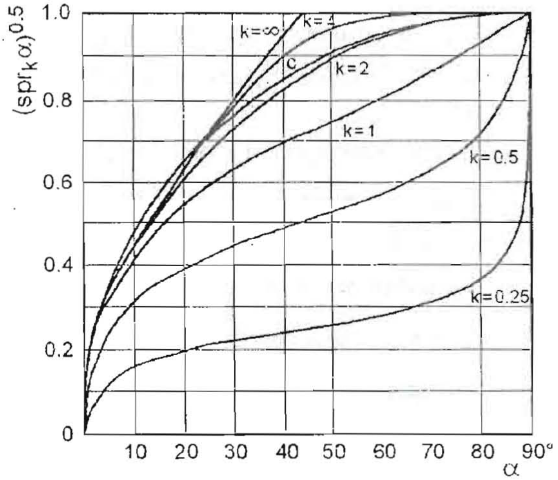

Fig. 11.3. The function $(\mathrm{spr}_k\alpha)^{0.5}$ for the k values according the Figure 11.1.

Regarding the Figures 11.1, 11.2 and 11.3 we have to mention that by variation of k in the entire interval of possible values, such as $0 \le k \le \infty$ coupled with the variation of p in the entire interval of possible values, such as $0 \le p \le \infty$, we can cover the entire surface of the first trigonometric quadrant in the domain $0^\circ \le \alpha \le 90^\circ$ and $0 \le spr_k\alpha \le 1.0$. We must stress the fact that any similarity of a sinusoidal periodic function $F(\alpha)$ with the function $(spr_k\alpha)^p$ when k and p have constant values, can be accepted when $F(\alpha)$ exactly concide $(spr_k\alpha)^p$ definite by the values of k and p respectively. In Physics and Technology we can meet with many such cases when they coincide, but in many situations $F(\alpha)$ can not be mathematically modeled to be equivalent with $(spr_k\alpha)^p$, where k and p have constant values. In these situations we can refuge to a mathematical modeling when k and p are variables. This modeling can't be done arbitrarily, but conform to some mathematical relations very well defined. About the power p, in this chapter we consider both situations, when the power p is constant and other examples when p is variable as a function of α.

11.2. The case when the paratrigonometric function is raised to some constant powers

We would like to mention that here, in order to simplify our analysis, in the Figure 11.1, 11.2 and 11.3, we used as a unit measure for the angle α, the old degree which is the case when the trigonometric circle has 360°. Also, we would like to underline another important fact is that the representation scale for 90° (corresponding to the trigonometric quadrant) on the abscissa axis was in this way chosen that this value would be equal with the dimension 1.0. The same dimensions would have to be equal to the maximum values of the functions $spr_k\alpha$ and $(spr_k\alpha)^p$ represented on the ordinate axis.

If we analyze the curves represented in the Figure 11.3 we see that one of the curves represented the function $(spr_k\alpha)^{0.5}$ has a form very closed to a circle arc developed on an angle of 90° (a quadrant of the circle). This thing is entirely confirmed after we traced the circle arc denoted by "c" with the center at the point with the coordinates $[\alpha = 90°; (spr_2\alpha)^{0.5} = 0]$ and having the radius 1.0. This circle arc is situated little more above the curve which represent the function $(spr_2\alpha)^{0.5}$, from where we conclude that it could be represented by the function $spr_2\alpha$ raised to the power p with little less than 0.5. We performed the calculations for more such values of the power p and the best result is obtained for $p = 0.445$. We make the observation that $spr_2\alpha$ is in fact the function $\sin\alpha$ from CT (Chapter 6) and thus for the circle arc "c" we can write the following equation:

$$y = (\sin\alpha)^{0.445}. \qquad (11.5)$$

In this equation y represents the ordinates of the points situated on the circle arc "c" and the abscissas are the values of the angle α. On the other hand, the equation of the circle arc "c" can be written expressing the y ordinate as a function of α in degree as such:

$$y = \{1-[1-(\alpha/90)]^2\}^{1/2}. \qquad (11.6)$$

Regarding the + or – signs that should appear in the relation (11.6), we will make the necessary discussion considering the rules applied in the CT.

From the equations (11.5) and (11.6) results the $\sin\alpha$ can be expressed by the following algebraic equation:

$$\sin\alpha = \{1-[1-(\alpha/90)]^2\}^{1.1}. \qquad (11.7)$$

This empirical relation (11.7) is very simple, but eminently is less precise then other algebraic relations for trigonometric functions, as the very well known algebraic series or "products" with an infinite number of terms, factors, respectively. It is known that these series or "products" give more precise values when the number of the terms and factors respectively are very large.

The relation (11.7) precision is fairly high, with errors up to 5.3%, if we refer to the values of the angle α comprise in the domain $3^\circ \le \alpha \le 90^\circ$. Surely then the use of the relation (11.5), which represent the paratrigonometric modeling of the circle, will bring us to the values for y affected by the same magnitude of error.

Everything what we have discussed above is referring to the circle arc "c", developed on the angle α comprised in the domain $0^\circ \le \alpha \le 90^\circ$. From the trigonometric point of view (supplementary angles) the relation (11.5) is thus valid and for the values of the angle α in the domain $0^\circ \le \alpha \le 180^\circ$. Also, from the algebraic point of view, by squaring the difference in the square parenthesis of these relations (11.6) and (11.7), they will become valid for the values of the angle α in this entire domain. For $180^\circ \le \alpha \le 360^\circ$, y of relation (11.5) has negative values. Since the angle α in this interval has negative values will imply that $\sin\alpha$ will be negative also and therefore to rise it to the power $p = 0.445$ is impossible.

In order to solve this problem we will write the relation (11.5) under the form:

$$y = \pm|\sin\alpha|^{0.445}. \qquad (11.8)$$

where $|\sin\alpha|$ is the absolute value of this function.

By the rules of the CT the sign + (plus) is used when the angle α is situated in the trigonometric quadrants I and II and the sign – (minus) is used when the angle α is situated in the trigonometric quadrants III and IV. We have to deal in the same way with the relations (11.7) and (11.8), but in this case for the values of α in the interval $180^\circ \le \alpha \le 360^\circ$, based on the very well known relations of the CT, where we replace angle α by the angle $\beta = \alpha - 180^\circ$. Thus we have:

- for $0^\circ \le \alpha \le 180^\circ$:

$$y = \{1-[1-(\alpha/90)]^2\}^{1/2} \qquad (11.9)$$

$$\sin\alpha = \{1-[1-(\alpha/90)]^2\}^{1.1} \qquad (11.10)$$

- for $180^\circ \le \alpha \le 360^\circ$:

$$y = -\{1-[1-(\beta/90)]^2\}^{1/2} \qquad (11.11)$$

$$\sin\alpha = -\{1-[1-(\beta/90)]^2\}^{1.1} \qquad (11.12)$$

where $\beta = \alpha - 180^\circ$.

For $\alpha > 360^\circ$ the function y in the relation (11.8) will have a periodic variation of the function $\sin\alpha$, as we very well know from the CT. The graphical representation of the function y for the three semi periods (540° respectively 3π rad.) is given in Figure 11.4.

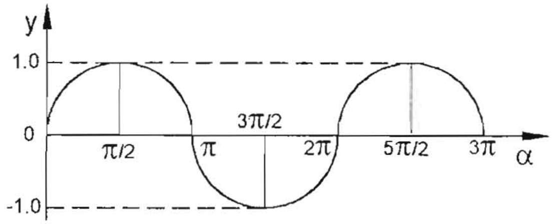

Fig. 11.4. The function y according the relations (11.9) and (11.11), for 3 semi periods (3π).

11.3. The case of a paratrigonometric function rise to a variable power

It is known that the periodic functions of the "sinusoidal" type are amply used in Electro-technology. In very many cases these functions cannot be assimilated by sinusoids and having different forms they are represented with some mathematical expressions sometimes more complex and with a graphical representation which very often results from experiments.

As an example related with a modeling possibility of such graphical representation by a paratrigonometric function raised to a variable power we will choose the case of a periodic function, which appears in the Technology of the Electrical Machines. Thus, we refer to the curve of the current i in the rotor of an electrical machine when the case where the magnetic field between the teeth limiting the slots is considered – [22], page 102, Fig. 76, a – We choose from this figure the case $U = 1.18 \cdot U_n$. We considered that the maximal value for i is equal with the unit and we traced it in coordinates $\left[\alpha; (spr_k\,\alpha)^p\right]$ for $0^\circ \le \alpha \le 90^\circ$ (Figure 11.5, curve "i").

If we also trace in this figure the curve which represent the paratrigonometric function $spr_k\,\alpha$ of order $k = 1$, thus we see that $spr_1\,\alpha$ has a trace very close with the curve which represents the current i on one and the other of its sides. We recall that the order $k = 1$ is characteristic to the Quadratic

Trigonometry (QT) – Chapter 6 – and thus, we can replace $spr_1\alpha$ by $sq\alpha$. The analyses of these two curves traces make us to deduce that the portion from the curve i function corresponding to some values of the angle α in the domain $0^\circ \le \alpha \le 47^\circ$ can be expressed by the function $sq\alpha$ raised to a supra unitary power ($p > 1$).

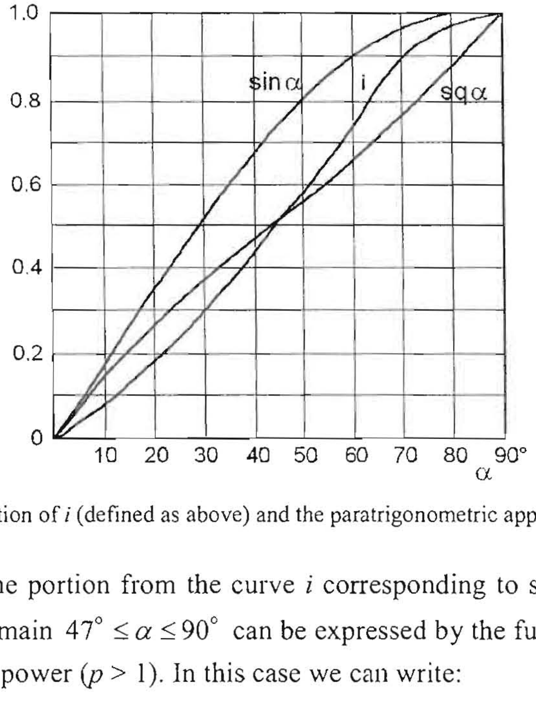

Fig. 11.5. Variation of *i* (defined as above) and the paratrigonometric approximations of *i*.

Contrary, the portion from the curve i corresponding to some values of the angle α in the domain $47^\circ \le \alpha \le 90^\circ$ can be expressed by the function $sq\alpha$ raised to a under unitary power ($p > 1$). In this case we can write:

$$i = (sq\alpha)^p \tag{11.13}$$

where p is, in this case, a variable value.

In order to determine the variation law of p as a function of α, for more of its values, we give p some diverse supra unitary values (for $\alpha < 47^\circ$) and under unitary values (for $\alpha > 47^\circ$) respectively, until the relation (11.13) is satisfied. This fact does not represent a problem when the computer is used. In the case of our example we performed these calculations for ten values of the angle α in the domain $0^\circ \le \alpha \le 90^\circ$. We mention that these ten values for the angle α were chosen in the interval $5^\circ \le \alpha \le 85^\circ$ since for $\alpha = 0^\circ$ we have $sq\alpha = 0$ and thus $i = 0$, and for $\alpha = 90^\circ$ we have $sq\alpha = 1$ and thus $i = 1$. The values of p obtained in this way are marked in a graph of $p = p(\alpha)$ in Figure 11.6 (with the reference to the ordinate axis on the left side).

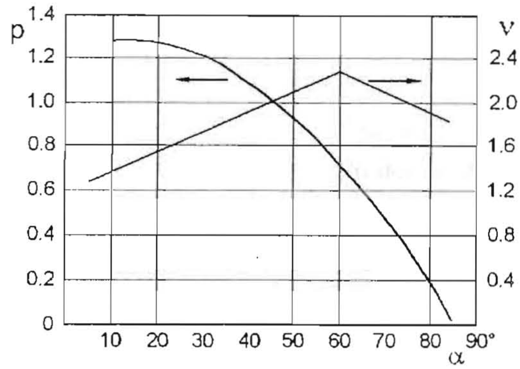

Fig. 11.6. Graphical representation of the function p according to 11.14 and of the function v according to 11.16.

Unifying the corresponding points we obtain the curve p which can be assimilated with a degree of precision sufficiently high with a parabola of second degree with the vertex at the point of coordinates ($\alpha = 10^\circ$; p = 1.3) and the ramifications developed towards the decreasing values of p. Having in view that this curve passes also through the point of the coordinates ($\alpha = 47^\circ$; p = 1.0), as we had shown above, we determined the definition elements of the parabola arriving to the following relation for it:

$$p = 1.3 - 2.245 \cdot 10^{-4} \cdot (\alpha - 10^\circ). \qquad (11.14)$$

Consequently it results that the variation curve of i as a function of α is mathematically modeled by the function $(sq\,\alpha)^p$ – see relation (11.13) – where p in its turn is a function of the angle α given by the relation (11.14). This mathematical model represents the variation curve of i conform [22] to a sufficient high precision, maximum error being of 5.6%.

Evidently that for $\alpha > 90^\circ$ the variation of i follows the trigonometric rules, starting from what we have shown above in regard with the corresponding function represented in the first trigonometric quadrant I.

The curve i can be modeled even if we admit that the function $(spr_2\,\alpha)^v$ which perform the modeling has v in function of the angle α, also. Since $spr_2\,\alpha = \sin\alpha$, we have:

$$i = (\sin\alpha)^v. \qquad (11.15)$$

Proceeding in the same way as above, we established values for corresponding to a sufficiency large number of the values for the angle α. Uniting the points of coordinates $(\alpha; v)$ we get with a sufficient precision these two concurrent lines in the point $(\alpha = 60^\circ; v = 2.3)$, represented in Figure 11.6 (referred to the ordinate axis on the right side).

These lines are represented by the equation:

$$v = 2.3 - 0.018 \cdot \left|60^\circ - \alpha\right| . \tag{11.16}$$

Calculating the values of the function i using the relations (11.15) and (11.16) we obtain the deviation of its values from the graph we intend to modeling of maximum 4.8%.

Using the examples discussed in this chapter we can see that there exists multiple possibilities to mathematically model some functions defined graphically by applying the paratrigonometric functions raised to some powers, generic written by $(spr_k\ \alpha)^p$, where k can have values in a very wide range and the power p in its turn can have diverse mathematical expressions under the form of some functions $p(\alpha)$.

11.4. Conclusions of Chapter 11

The following important conclusions summarize what we have discussed in these above-mentioned chapters:

11.4.1. The paratrigonometric functions, especially the function $spr_k\ \alpha$, have multiple capabilities to model functions graphically defined which are frequently met in Physics and Technology, respectively. This thing is due the fact that the "order" k of the paratrigonometric function can have infinitely many values. These capabilities of the paratrigonometric function $spr_k\ \alpha$ can be more multiplied if the corresponding function is raised to a power p. In this case we have the function $(spr_k\ \alpha)^p$.

11.4.2. Many graphically defined functions and some algebraically defined functions can be modeled by paratrigonometric functions raised to a constant power. Such a case is represented in the modeling of the circle with a high precision.

11.4.3. A large possibility to model some functions graphically defined which are frequently met in Physics and in Technology is to use the paratrigonometric functions raised to some power which themselves are defined as functions of the angle α. In Subchapter 11.3 we discussed the case of a function graphically defined.

12. PARATRIGONOMETRIC FUNCTIONS RELATIVE TO THE FINITE SPIRALS AS THE BASIC TRIGONOMETRIC FIGURES

12.1. Introduction

In Chapter 6 we analyzed the paratrigonometric functions represented in the Chartesian coordinates and we showed the connection between these functions and The Basic Trigonometric Figures (BTFs). These last ones in their turn were also represented in the Chartesian coordinates.

We also, recall that the fundamental relations in the Paratrigonometry are the following:

$$|spr_k\alpha|^k + |cpr_k\alpha|^k = 1 \tag{12.1}$$

$$tpr_k\,\alpha = tg\,\alpha \tag{12.2}$$

where $spr_k\,\alpha$ is "the paratrigonometric sine of order k of the angle α", $cpr_k\,\alpha$ is "the paratrigonometric cosine of order k of the angle α" and $tpr_k\,\alpha$ is "the paratrigonometric tangent of order k of the angle α". The order k can have values in the domain $0 \le k \le \infty$. Important particular cases are represented when $k = 2$ (The Classical Trigonometry – CT) and $k = 1$ (The Quadratic Trigonometry – QT).

The Basic Trigonometric Figures (BTFs) of the corresponding paratrigonometric functions are in their turn expressed in Chartesian coordinates by the relation:

$$|y|^k + |x|^k = 1\,. \tag{12.3}$$

In the CT ($k = 2$) the corresponding BTF is a circle having its radius $R = 1$. In the QT ($k = 1$) BTF is a rhombus with all its angles being right angles, which is inscribed in a circle of the radius $R = 1$. For any other values of k the BTFs are "rhombuses" with curved sides, which are convex for $1 < k \le \infty$ and concave for $0 \le k < 1$.

All of the BTFs presently studied in regard with the PRT are symmetric with the Chartesian coordinate axis Ox – Oz.

We intend to study further these non-symmetric BTFs corresponding to these axes. We will bring in our analyse those BTFs of spiral form, which develop between the coordinate point ($x = 1$; $y = 0$), for $\alpha = 0$ and the point of the coordinates ($x = 0$; $y = 0$), for $\alpha = 2 \cdot K \cdot \pi$, K an integer ($K = 1, 2, \ldots$, etc.).

Evidently, for the best representation of a spiral we are going to use the polar coordinates.

In Chapter 10 we used the polar coordinates to represent some paratrigonometric functions.

In what follows, we will analyze some paratrigonometric functions which are related with these BTFs under the form of finite spirals, that is that the spirals start and end in very well defined points in the coordinates system, as we have shown above.

12.2. Archimedean Spiral, Logarithmic Spiral and Parabolic Spiral having BTFs role in the Paratrigonometry

The classical mathematical expression for the Archimedean Spiral in the polar coordinates is:

$$\rho = c \cdot \alpha \tag{12.4}$$

where ρ is the polar radius, α is the angle formed by the polar radius with the polar axis Op (see Fig. 12.1) and c is a constant. In other words, the polar radius varies directly proportional with the angle α.

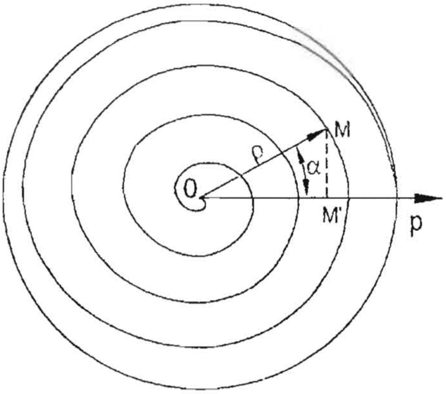

Fig. 12.1. The Archimedean Spiral

We can see that the Archimedean Spiral, expressed in this way – see (12.4), starts from the pole O (for $\alpha = 0$, $\rho = 0$) and tends towards infinity (for $\alpha = \infty$, $\rho = \infty$).

In order to establish the expression for the polar radius which decreases from $\rho = 1$ (for $\alpha = 0$) down to $\rho = 0$ (for $\alpha = 2 \cdot K \cdot \pi$), as we have shown above, we will use for the corresponding spiral (which we name "finite") the relation:

$$\rho = 1 - (\alpha / 2\pi \cdot n) \tag{12.5}$$

where n represents the number of the complete spires (by 2π rad. each) developed between $\rho = 1$ and $\rho = 0$. In Fig. 12.1 a such spiral is represented for which $n = 4$.

This spiral is developing in a trigonometric direction from $\rho=1$ (for $\alpha=0$) to $\rho=0$ (for $\alpha=8\pi$). For $\alpha>8\pi$ the polar radius ρ becomes negative and this representation does not have any sense.

Another very well known spiral in Mathematics is the Logarithmic Spiral. This one when the angle α increases in a trigonometric sense is represented by the relation:

$$\rho=c\cdot e^{-m\alpha} \tag{12.6}$$

where c and m are constants greater then 0 (zero), and α is the angle formed by the polar radius ρ with the polar radius Op, as we previously had shown. Since we set the condition that for $\alpha=0$ to have $\rho=1$, from the relation (12.6) results that $c=1$ and the relation (12.6) results that $c=1$ and the relation (12.6) becomes:

$$\rho=e^{-m\alpha}\,. \tag{12.7}$$

The polar radius ρ tends to 0 (zero) when α tends to $+\infty$. In other way saying, the pole O is the asymptotic pole where the spiral is approaching more and more for α increasing to $+\infty$, but O is never touched by the spiral. This "comes" from $\rho=\infty$ when $\alpha=-\infty$ and passes through the point ($\alpha=0;\rho=1$), towards the pole O, which can be theoretically touched for $\alpha=+\infty$.

Compared with the previous situation (Archimedean Spiral) we accept that in function of the number of the spirals (of 2π rad. each) from which we establish that the entire spiral is formed, the value of ρ is very small and we denote it by $\Delta\rho$. In this case we have:

$$\Delta\rho=e^{-2\pi\cdot n\cdot m}\,. \tag{12.8}$$

In order to determine m we take logarithm in the relation (12.8) and we obtain:

$$m=-(\ln\Delta\rho)/2\pi\cdot n\,. \tag{12.9}$$

Because in every case $\Delta\rho<1$ then the values of m will be positive.

Accepting $n=4$, as in the Archimedean Spiral case, we have:

$$m=-(\ln\Delta\rho)/8\pi\,. \tag{12.10}$$

Considering for example, $\Delta\rho=0.025$ we obtain $m=0.147$.

A Logarithmic Spiral conform the relation (12.7) and having four spires ($n=4$) and thus $m=0.147$, is represented in the Figure 12.2.

A spiral with a similar form to the Archimedean Spiral is the one represented by the following equation:

$$\rho=a\cdot\alpha^{p}+b \tag{12.11}$$

where a, b and p are constant values.

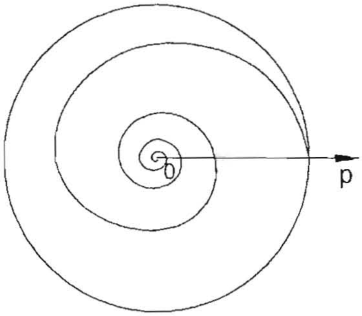

Fig. 12.2. The Logarithmic Spiral.

Because the variable α is raised to a power (p), we name the corresponding curve to be the "Parabolic Spiral".

If in the relation (12.11) we make $\alpha = 0$, then we get $b = 1$ (in order to have $\rho = 1$). If for $\alpha = 2\pi \cdot n$, we accept $\rho = 0$, then $a = -(1/2\pi \cdot n)^p$. Thus the equation (12.11) becomes:

$$\rho = 1 - (\alpha / 2\pi \cdot n)^p . \tag{12.12}$$

For $p = 1$ the relation (12.12) is identical with the relation (12.5). If we accept $n = 4$, as above, the relation (12.12) becomes:

$$\rho = 1 - (0.0398 \cdot \alpha)^p . \tag{12.13}$$

The value for p can be chosen in a such a way that the curve of the function $\rho(\alpha)$ can mathematically model in a very accurate way a specific fenomenon (in Physics, for example) which can be represented by the relation (12.13).

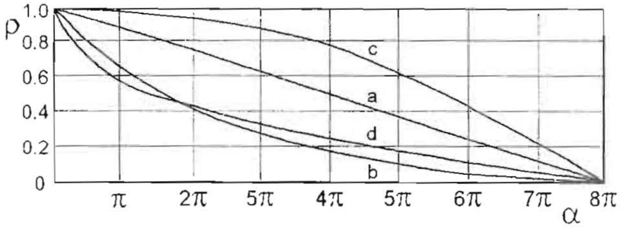

Fig. 12.3. The vector radius as a function of the angle α, $\rho(\alpha)$ for diverse spiral.

In the Figure 12.3 we represent the curves for the function $\rho(\alpha)$ expressed by the above relations for $n = 4$.

Thus:

– the curve a (straight line), relation (12.5) for $n = 4$ and the relation (12.13) for $p = 1$, respectively;

– the curve b, the relation (12.7) for $m = 0.147$ corresponding to $\Delta\rho = 0.025$ – see relation (12.10);

– the curve *c*, relation (12.13) for $p = 2$;
– the curve *d*, relation (12.13) for $p = 0.4$.

The value $p = 0.4$ above was chosen by trying, so that the form of the curve *d* to be the closest to the form of the curve *b*.

To some Basic Trigonometric Functions (BTF) having spiral forms, analyzed above, exist corresponding specific trigonometric functions, and this will be discussed in the next chapter.

12.3. The Paratrigonometric Spiral Functions

We call the Paratrigonometric Spiral Functions (PSFs) those paratrigonometric functions, which are referred to BTFs with a spiral form. We will analyze those PSFs, which are correlated with the spirals presented in the previous chapter as BTFs.

We denote by $Sps\,\alpha$ the function "Spiral Paratrigonometric Sinus of the angle α", with $Cps\,\alpha$ the function "Spiral Paratrigonometric Cosine of the angle α" and with $Tps\,\alpha$ the function "Spiral Paratrigonometric Tangent of the angle α".

Referring to the Figure 12.1, we see that $Sps\,\alpha$ is equal with the quotient between the magnitude of the line segment MM' and the vector radius ρ. The function $Cps\,\alpha$ is equal with the quotient between the magnitude of the line segment OM' and the vector radius ρ. Between these functions there are the following relations:

$$(Sps\,\alpha)^2 + (Cps\,\alpha)^2 = \rho^2 \tag{12.14}$$

$$Sps\,\alpha \,/\, Cps\,\alpha = Tps\,\alpha = tg\,\alpha\,. \tag{12.15}$$

We see that these relations are similar with the fundamental relations from the Paratrigonometry, which are in relation with the BTFs symmetric to the coordinate axis Ox – Oy [6]. There is also, a similarity of the relation (12.14) with the fundamental relation from the Classical Trigonometry (CT):

$$\sin^2\alpha + \cos^2\alpha = 1\,. \tag{12.16}$$

The distinction between these two relations consists from the fact that, in this case which we are analyzing now, in the right side of the equality instead of a constant (number 1) appears ρ, which is an algebraic function of α.

Also, from Figure 12.1 we observe that the functions $Sps\,\alpha$ and $Cps\,\alpha$ can be expressed by the functions $\sin\alpha$ and $\cos\alpha$ (and ρ) in this way:

$$Sps\,\alpha = \rho\cdot\sin\alpha \tag{12.17}$$

$$Cps\,\alpha = \rho\cdot\cos\alpha\,. \tag{12.18}$$

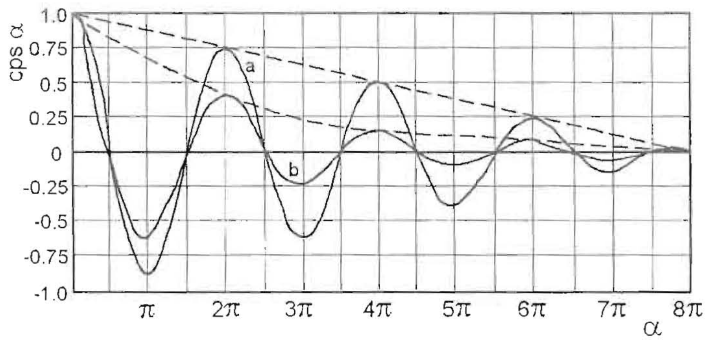

Fig. 12.4. The functions $Cps_A\alpha$ and $Cps_L\alpha$.

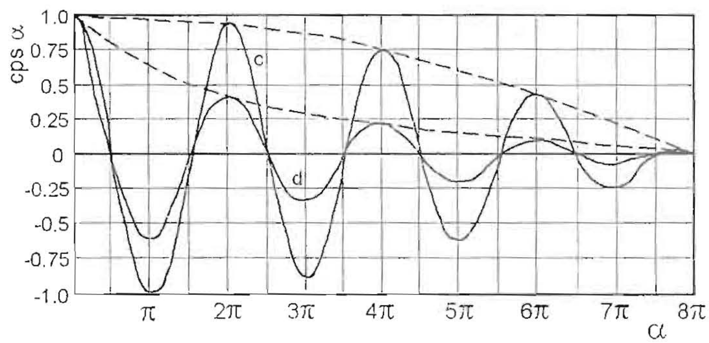

Fig. 12.5. The functions $Cps_P\alpha$.

In the Figures 12.4 and 12.5 the function $Cps\,\alpha$ is represented for the following situations:

- in Figure 12.4:
  - *a* – for the Archimedean Spiral, as a BTF (see Figure 12.1);
  - *b* – for the Logarithmic Spiral, as a BTF (see Figure 12.2);
- in Figure 12.5:
  - *c* – for the Parabolic Spiral, (with $p = 2$) as a BTF;
  - *d* – for the Parabolic Spiral, (with $p = 0.4$) as a BTF.

It is interesting to remark that if we develop the relation (12.18), using for p the relation (12.7) which is characteristic to the Logarithmic Spiral, we obtain:

$$Cps_L\,\alpha = e^{-m\cdot\alpha}\cdot\cos\alpha\,. \tag{12.19}$$

We used the notation $Cps_L\,\alpha$ in order to remark the fact that the $Cps\,\alpha$ is to the reference for a Logarithmic Spiral (the index L), as BTF: by analogy, we will also use the notations $Cps_A\,\alpha$ when we refer to an Archimedean Spiral (the index A), as BTS and respectively $Cps_P\,\alpha$ when we refer to a Parabolic Spiral (the index P), as BTF.

The relation (12.19) is exactly the equation for the Amortized Oscillations from Physics and Mechanics respectively if we refer to the Mechanical Systems [23], [24].

It is known that, the mathematical expression which characterizes the Amortized Vibrations is:

$$x = x_0 \cdot \exp(-h \cdot t) \cdot \cos(\omega \cdot t + \varphi) \qquad (12.20)$$

where x is the elongation, x_0 is the variations amplitude, h is the amortization factor, t is the time, ω is the pulsation (circular frequency), φ is the initial phase (diphase). For simplification if we accept $x_0 = 1$ and $\varphi = 0$, we obtain the relation:

$$x = \exp(-h \cdot t) \cdot \cos(\omega \cdot t). \qquad (12.21)$$

This relation is similar with the relation (12.19), if we consider $x = Cps_L\, \alpha$, $h \cdot t = m \cdot \alpha$ and $\omega \cdot t = \alpha$, thus $h / m = \omega$. In another words saying $Cps_L\, \alpha$ can represent an amortized vibration, in the case when the physical characteristics of the vibration (h, t and ω) are adequately in the relation (12.19).

Coming back to the Figures 12.4 and 12.5, we see that in the Figure 12.4 "the enveloping curve" of the curve *b* represents the graphical expression of the relation (12.7). This curve was traced for positive values only of the function $Cps_L\, \alpha$. It is similar with the curve *b* of Figure 12.3. In the same way "the enveloping curve" of the curve *d* of Figure 12.5 represents the graphical expression of the relation (12.13), for $p = 0.4$ and it is similar with the curve *d* of Figure 12.3.

If we choose adequately the value of p, as we have shown before, the curves *b* and *d* of Figure 3 are looking very close alike, even if they refer to the different BTFs namely, the Logarithmic Spiral and Parabolic Spiral, respectively.

12.4. Conclusions of Chapter 12

From what we have shown in the previous chapters, we can take the following important conclusions:

12.4.1. In the Paratrigonometry (Chapter 6) beside the Symmetric Basic Trigonometric Figures (BTFs) we can use the asymmetric BTFs with respect to the coordinate axis Ox – Oy. In this paper we analyzed as BTFs the following finite spirals, developed in the trigonometric sense with the values of the angle α between $\alpha = 0$ up to $\alpha = 2 \cdot K \cdot \pi$ (where K is a positive integer number):

- Archimedean Spiral;
- Logarithmic Spiral;
- The spiral which we named "Parabolic Spiral".

The mathematical modeling of these spirals in this chapter was done in the simplest possible manner such as by representing them in the polar coordinates.

The spirals are developing from the polar radius $\rho=1$ towards $\rho=0$, even to $\rho=0$ or tending to this value (for the Logarithmic Spiral).

12.4.2. The Paratrigonometric Functions corresponding to the BTFs mentioned above (Spirals) denoted by $Sps\,\alpha$, $Cps\,\alpha$, etc., are expressed by the product of the vector radius function which characterize the corresponding spiral $\rho(\alpha)$, and the trigonometric functions of the Classical Trigonometry (CT), $\sin\alpha$, $\cos\alpha$, etc.

12.4.3. The function $Cps_L\,\alpha$ referring to the Logarithmic Spiral (from where we have the index L) as BTF, coincide with the mathematical expression of the elongation in the case of the amortized mechanical vibrations.

13. POSSIBILITIES TO REPRESENT SOME SYMMETRIC FUNCTIONS BY UNIFIED EQUATIONS AND SOME OF THEIR APPLICATIONS IN THE PARATRIGONOMETRY

13.1. Introduction

Many times in practice, especially in Technology, we meet situations when some graphical representations as contours of mechanical parts etc. consist in putting together symmetrically of some curves segments which can't be mathematically modeled by a single equation but by two distinct equations. These curves together are not represented by monotonic functions [18, 19] but they are "fractured" at their contact point (Chapter 3) and in the Classical Mathematics the attached functions to these curves are represented by distinct equations. Examples of this kind are the tooted wheel profiles and the profiles of some cams, etc.

In what follows, we find a way to represent these two equations which commonly are represented by two algebraic symmetric functions, by a unified equation. For example, we will analyze some of such cases, starting with the simplest one, namely the case of a fractured line segment represented by the equation $y = a \cdot x$ (a const.) in the domain $0 \leq x < \lambda$, accepted as the "basic function" (the segment *OM* of Figure 13.1). The "Symmetric Function" of the respective basic function is characteristic to the segment *MN* (corresponding to the domain $\lambda \leq x \leq 2\lambda$) of Figure 13.1.

With this goal in our mind, we will analyze this first case as well as some others in which intervene equations of superior degree and also some paratrigonometric functions.

13.2. The case of the basic function $y = a \cdot x$.

From the beginning we mention that in general the symmetric functions have for symmetric axis the ordonate Oy axis [18]. In order to simplify what follows in our analysis we will accept as the symmetry axis a line parallel with Oy axis situated at a distance λ from it.

In Figure 13.1 is represented the basic function (denoted by y_b), characteristic to the segment *OM* (corresponding to the interval $0 \leq x < \lambda$) and the symmetric function (denoted by y_s), characteristic to the line segment *MN* (corresponding to the interval $\lambda \le x \le 2\lambda$), as we have shown above.

Thus we have:

$$y_b = a \cdot x_b . \tag{13.1}$$

If we consider that it for $x_b = \lambda$ we have $y_b = m$ the equation (13.1) becomes:

$$y_b = (m / \lambda) \cdot x_b . \tag{13.2}$$

We mention that by x_b we denoted the current abscises for the functions represented by the equations (13.1) and (13.2) respectively and thus is referred to the segment *OM*.

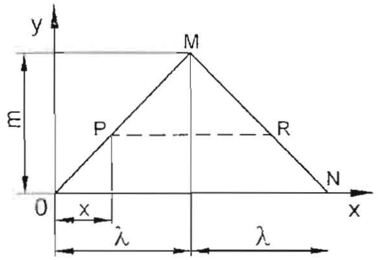

Fig. 13.1. Graphical representation of some line segments *OM* and *MN* symmetrically situated.

As we have shown, the line segment *MN* is symmetric with *OM*, with respect to the vertical line which passes trough *M*. The general form of the line equation where the segment *MN* belongs is:

$$y_s = b \cdot x_s + c . \tag{13.3}$$

We mention that by x_s we denoted the current variable referred to the segment *MN*. The constants *b* and *c* can be determined using the values for y_s and x_s at the limits of the interval $\lambda \le x \le 2\lambda$; thus for $x_s = \lambda$ we have $y_s = m$, and for $x_s = 2\lambda$ we have $y_s = 0$. Introducing the above values in the equation (13.3), we obtain $b = -(m / \lambda)$ and $c = 2m$ and the respective equation becomes:

$$y_s = 2m - (m / \lambda) \cdot x_s . \tag{13.4}$$

The equations (13.2) and (13.4) for the basic function and the symmetric function respectively are different at the first glance, they can't be written under an unified form. In order to do that we see that in Figure 13.1 the value of y_b at the point *P* is equal with the value of y_s at the point *R*. Thus we have:

$$y_{b(P)} = y_{s(R)} . \tag{13.5}$$

In the other side, the abscise values of the two cases are different, but between them there exist the relation

$$x_{s(R)} = 2 \cdot \lambda - x_{b(P)} . \tag{13.6}$$

This means that these two equations (13.2) and (13.4) could be unified under a general form:

$$y = -(m / \lambda) \cdot x + 2 \cdot m . \tag{13.7}$$

If we can find a way that for the interval $0 \le x < \lambda$ the sign of the factor (m / λ) could change from – (minus) to + (plus) and to dismiss the term $2m$.

This think is possible if we introduce the factors A and B which we name "Binary Operators" motive which we will further explain. These operators are given by the following relations:

$$A = (x - \lambda) / |x - \lambda| \tag{13.8}$$

$$B = \left[(x - \lambda) + |x - \lambda| \right] / 2|x - \lambda| . \tag{13.9}$$

In the above relations $(x - \lambda)$ represents the algebraic value of the difference between x and λ, and $|x - \lambda|$ represents the absolute value of this difference. We can see that for $0 \le x < \lambda$, we have $A = -1$, and for $\lambda \le x < 2\lambda$ we have $A = +1$. Now, what happen when $x = \lambda$? In some Mathematical Works [18] is shown that a function of x, as it is A, does not have a limit for $x = \lambda$. In other words saying, for $(x - \lambda - \varepsilon)$, where ε is a positive value infinitely small, we have $A = -1$ and for $(x - \lambda + \varepsilon)$ we have A = +1. On the other side we observe that if in the relation (13.8) we make $x = \lambda$, for A appears an indetermination $A = 0/0$. If we apply l'Hôspital Rule to eliminate the indetermination for the point M, we will have $d(x - \lambda) / d|x - \lambda| = 1$ and thus for this point the operator $A = +1$. With another way saying the point M belongs to the segment MN, and the segment OM approaches to the point M up to its immediate neighborhood. Thus we can say that $A = -1$ for $0 \le x < \lambda$ and $A = +1$ for $\lambda \le x < 2\lambda$. For this reason and because of what we have mentioned before, we used these corresponding limits.

Applying the same reasoning from above for the operator B case it results that for $0 \le x < \lambda$ we have $B = 0$, and for $\lambda \le x < 2\lambda$ we have $B = +1$.

We named A and B as "Binary Operators" since for these two value intervals of x above mentioned, they change their value from -1 to $+1$ (for operator A case) and from 0 to +1 (for operator B case) or inversely (see Table 13.1 in continuation).

These facts being established in order to give to the formula (13.7) a unique general form for the two analyzed symmetric functions, we use in a proper way these two operators A and B and we obtain the equation:

$$y = -A \cdot (m / \lambda) \cdot x + 2 \cdot B \cdot m . \tag{13.10}$$

For the interval $0 \le x < \lambda$, we have $A = -1$ and $B = 0$ and the equation (13.10) becomes the equation (13.2). For the interval $\lambda \le x < 2\lambda$ we have $A = 1$ and $B = 1$ and the equation (13.10) becomes the equation (13.7).

These operators mentioned above (*A* or *B*) were used by the authors in the Chapter 9.

13.3. The case of the circle segments (quarters)

In Figure 13.2 we represent as the basic function graph the quarter of the circle *OM*. The graph of the symmetric function is the quarter of the circle *M*.

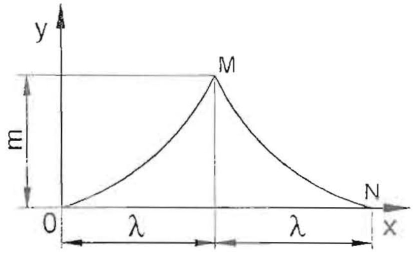

Fig. 13.2. Circle quadrants *OM* and *MN* symmetrically situated.

The equations of the corresponding functions are:

$$y_b = \lambda - \left(\lambda^2 - x^2\right)^{1/2} \tag{13.11}$$

$$y_s = \lambda - \left[\lambda^2 - (2\lambda - x)^2\right]^{1/2}. \tag{13.12}$$

Proceeding as in the previous case, discussed in Subchapter 13.2, we obtain this unique equation

$$y = \lambda - \left[\lambda^2 - x^2 - 4B\lambda(\lambda - x)\right]^{1/2}. \tag{13.13}$$

For $0 \le x < \lambda$ we have $B = 0$ and the equation (13.13) becomes the equation (13.11) and for $\lambda \le x < 2\lambda$ we have $B = 1$ and the equation (13.13) becomes the equation (13.12).

13.4. The hyperbolas case

We accept hyperbola as basic function represented by the equation:

$$y_b = \left(a \cdot x^2 + b \cdot x\right)^{1/2}. \tag{13.14}$$

This hyperbola passes through the origin and does not have in equation (13.14) any constant unassociated with the variable x.

Beside the point with the coordinates $(x = 0;\ y_b = 0 \cdot c)$, we consider that the hyperbola also passes through the point of the coordinates $(x = 0.5 \cdot \lambda;\ y_b = 0.6 \cdot m)$ – see Figure 13.3, curve *OM*.

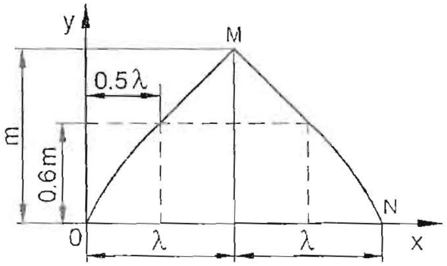

Fig. 13.3. Hyperbola segments *OM* and *MN* symmetrically situated.

In these conditions the equation (13.14) becomes:

$$y_b = \left[0.56 \cdot (m/\lambda)^2 \cdot x^2 + 0.44 \cdot \left(m^2/\lambda\right) \cdot x\right]^{1/2}. \tag{13.15}$$

Proceeding, as in the previous cases, now for the segment *MN* of the hyperbola, symmetric with the segment *OM*, we have the equation

$$y_s = \left[0.56 \cdot (m/\lambda)^2 \cdot x^2 - 2.68 \cdot \left(m^2/\lambda\right) \cdot x + 3.12 \cdot m^2\right]^{1/2}. \tag{13.16}$$

In order to make the unification of equations (13.15) and (13.16) possible we write the equation (13.15) under the form:

$$y_b = \left[0.56 \cdot (m/\lambda)^2 \cdot x^2 - 2.68 \cdot (m^2/\lambda) \cdot x + 3.12 \cdot (m^2 \lambda) \cdot x\right]^{1/2}. \tag{13.17}$$

We see that in the equations (13.16) and (13.17) the first two terms are identical. Thus, we need operators of type A and B from above to obtain solutions to make active the third term from each of these equations.

In this way we can write the unified equation for the symmetric functions afferent to the hyperbolic segments *OM* and *MN* as:

$$\begin{aligned} y = \Big[& 0.56 \cdot (m/\lambda)^2 \cdot x^2 - 2.68 \cdot (m^2/\lambda) \cdot x + \\ & + 3.12 \cdot (m^2 \lambda) \cdot (B - A) \cdot x + 3.12 \cdot B \cdot m^2 \Big]^{1/2} \end{aligned} \tag{13.18}$$

For the interval $0 \le x < \lambda$, when $B = 0$ and $A = -1$ thus $(B - A) = 1$, the equation (13.18) becomes the equation (13.15). For the interval $\lambda \le x < 2\lambda$, when $B = 1$ and $(B - A) = 0$, the equation (13.18) becomes the equation (13.16).

In Table 13.1 we give synthetically all the possibilities in the change of the algebraic signor or the annihilation of the binary operators A and B which can appear alone or together in our present analysis.

Table 13.1

The values of the Binary Operators A and B

The interval	The operator					
	A	$-A$	B	$-B$	$(A-B)$	$(B-A)$
$0 \le x < \lambda$	-1	$+1$	0	0	-1	$+1$
$\lambda \le x \le 2\lambda$	$+1$	-1	$+1$	-1	0	0

13.5. The case of the cubic parabolas

We accept as a basic function a cubic parabola having as equation

$$y_b = a \cdot x^3 + b \tag{13.19}$$

and also we accept that for $x = 0$; $y_b = 0.1 \cdot m$, thus $b = 0.1 \cdot m$. For $x = \lambda$ we will have

$$y_b = a \cdot \lambda^3 + 0.1 \cdot m \tag{13.20}$$

from where $a = 0.9 \cdot m / \lambda^3$ and thus the equation (13.19) becomes

$$y_b = (0.9 \cdot m / \lambda^3) \cdot x^3 + 0.1 \cdot m. \tag{13.21}$$

For the interval $0 \le x < \lambda$ to this equations is corresponds the segment QM of Figure 13.4.

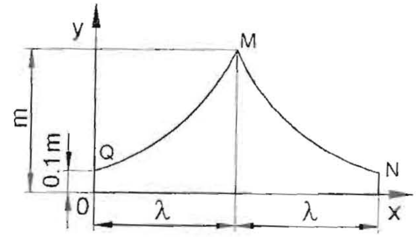

Fig. 13.4. Cubic parabola segments QM and MN symmetrically situated.

The symmetric parabolic segment MN (in the interval $\lambda \le x < 2\lambda$) is represented by the equation

$$y_s = (0.9 \cdot m / \lambda^3) \cdot (2\lambda - x)^3 + 0.1 \cdot m. \tag{13.22}$$

Comparing the equations (13.22) and (13.21) and considering the values of the operators A and B from Table 13.1, it follows that the unified equation of the two symmetric parabolas is

$$y = (0.9 \cdot m / \lambda^3) \cdot (2B\lambda - Ax)^3 + 0.1 \cdot m . \tag{13.23}$$

For the interval $0 \le x < \lambda$, we have $A = -1$ and $B = 0$ and thus the equation (13.23) becomes equation (13.21). For the interval $\lambda \le x \le 2\lambda$, we have $A = 1$ and $B = 1$ and thus the equation (13.23) becomes equation (13.22).

13.6. The case of the exponential functions

In this chapter we will analyze two symmetric functions of which the graphical representation is distinct from those of the previous chapters, in the way that in the basic function the value of y decreases at the same time when the value of x increases.

Thus we accept the case of two exponential functions when the basic function has the form:

$$y_b = -a \cdot e^x + b . \tag{13.24}$$

Now we impose the following conditions: for $x = 0$ we have $y_b = m$, and for $x = \lambda$ we have $y_b = 0$.

If we introduce these constants a and b in the equation (13.24) we obtain:

$$y_b = [m / (e^\lambda - 1)] \cdot (e^\lambda - e^x) . \tag{13.25}$$

This equation is valid for the segment QM of Figure 13.5.

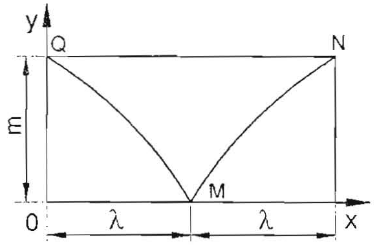

Fig. 13.5. Exponential Functions segments *QM* and *MN* symmetrically situated.

Using a similar reasoning as above, for the segment MN (symmetric with QM) we obtain the equation:

$$y_s = [m / (e^{-\lambda} - 1)] \cdot [e^{(\lambda - x)} - 1] . \tag{13.26}$$

Comparing the equations (13.25) and (13.26) and using the operators A and B, we obtain the following unified equation for the two symmetric functions:

$$y = [m / (e^{-A\lambda} - 1)] \cdot [e^{(\lambda - Bx)} - e^{(B-A)x}]. \qquad (13.27)$$

We see that for the interval $0 \le x < \lambda$, the equation (13.27) becomes equation (13.25) and for the interval $\lambda \le x \le 2\lambda$, the equation (13.27) becomes equation (13.26).

13.7. An application in the Paratrigonometry

In chapter 11 we analyzed the case of a representation by a paratrigonometric function (raised to a variable power) of the electrical current i in the rotor of an electrical machine for some conditions of its operation.

In this chapter we give a mathematical model for a portion of this curve with the values of the angle α (rotation angle of the rotor) comprised in the domain $0^\circ \le \alpha < 90^\circ$. The corresponding curve is represented in Figure 13.6 and its portion in the domain mentioned above is comprised between the points O and M, in order to enter in the same notation system used in this paper. In other words, we represent the basic curve i_b by the curve OM.

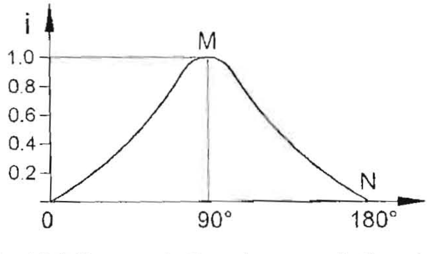

Fig. 13.6. Symmetric Paratrigonometric Function.

For simplification, i was represented in a such a way that its maximum value (for $\alpha = 90^\circ$) is 1 (one).

From Chapter 11 we remember that i and i_b respectively, in our case is given by the relation

$$i_b = (sq\,\alpha)^p \qquad (13.28)$$

where $sq\,\alpha$ is the function "sinus paratrigonometric of order $k = 1$ of the angle α", which is the same thing as "sinus quadratic of the angle α". In its turn, p is a function of the angle α, given by the relation

$$p = 1.3 - 2.245 \cdot 10^{-4} \cdot (\alpha - 10^\circ). \qquad (13.29)$$

Since, as we have shown, the relation i_b is valid for values of the angle α comprised in the domain $0^\circ \le \alpha < 90^\circ$, in the Chapter 11 was mentioned that, for the domain $90^\circ \le \alpha \le 180^\circ$, the form of the curve is obtained applying the trigonometric rules for the supplementary angles.

In what follows, for the domain $90^\circ \le \alpha \le 180^\circ$, we will solve this problem applying the method developed in the previous subchapters.

Thus, for the portion *MN* of the curve in the function *i*, that is for the symmetric portion *OM* we will have the relation

$$i_s = [sq(180^\circ - \alpha)]^\sigma . \tag{13.30}$$

where

$$\sigma = 1.3 - 2.245 \cdot 10^{-4} \cdot (170^\circ - \alpha) . \tag{13.31}$$

Using the binary operators of the type *A* and *B* mentioned above, we can write this unified relation for *i*, which is valid for the entire interval $0^\circ \le \alpha \le 180^\circ$, under this form:

$$i = (B^\circ - A^\circ) \cdot i_b + B^\circ \cdot i_s \tag{13.32}$$

or using the relation (13.28) and (13.30):

$$i = (B^\circ - A^\circ) \cdot (sq\,\alpha)^p + B^\circ \cdot [sq(180 - \alpha)]^\sigma \tag{13.33}$$

where A° and B° are the binary operators *A* and *B*, conform the relations (13.8) and (13.9) from above where the measure x was replaced with α [°], and the measure λ was replaced with 90° compatible with our case.

For the interval $0^\circ \le \alpha < 90^\circ$ we will have $(B^\circ - A^\circ) = +1$ and $B^\circ = 0$ (see Table 13.1), and thus the relation (13.33) becomes the relation (13.28). For the interval $90^\circ \le \alpha \le 180^\circ$ we will have $(B^\circ - A^\circ) = 0$ and $B^\circ = +1$, and thus the relation (13.33) becomes the relation (13.30).

13.8. Conclusions of Chapter 13

From what we have discussed above we can mention some important conclusions:

13.8.1. There exist some situations when some symmetric functions appear and they have graphical representations symmetric with respect to some symmetric axis. These functions were before represented with distinct equations.

13.8.2. Introducing some special mathematical operators named "binary operators" we can relatively easy formulate unified equations for the symmetric functions. The different values of the respective operators (+1, –1 or 0) introduced in the unified equations will particularize them, making them valid for the "Basic function" respectively for the "Symmetric function".

13.8.3. We analyzed for example, functions of degree 1, 2 and 3 as well as exponential functions and also a paratrigonometric function.

14. A SIMPLIFIED ALTERNATIVE TO DEVELOP PERIODIC FUNCTIONS IN TRIGONOMETRIC FOURIER SERIES

14.1. Introduction

We know the fact that the periodic functions having a constant period which can't be expressed by a unique equation can be brought to the situation to be expressed by an unique equation using the Fourier Trigonometric Series [18], [20]. Thus, for example, a periodic succession of the line segments graphically represented in Figure 14.1 (named rectangular function) is mathematically expressed by the following equations:

– for $0 < x < \pi$

$$y = b \tag{14.1}$$

– for $\pi < x < 2\pi$

$$y = -b \tag{14.2}$$

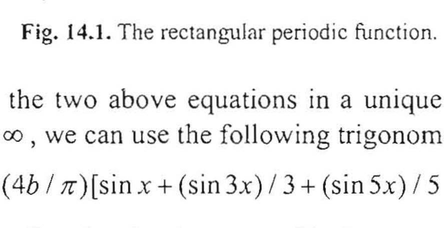

Fig. 14.1. The rectangular periodic function.

In order to put the two above equations in a unique equation valid for the entire domain $0 < x < \infty$, we can use the following trigonometric series:

$$y = (4b / \pi)[\sin x + (\sin 3x) / 3 + (\sin 5x) / 5 + ...]. \tag{14.3}$$

Also, a periodic function having a graphical representation of the form of "saw teeth" (triangular function), as in Figure 14.2, can be expressed by the equation:

$$y = (8b / \pi^2)[\sin x - (\sin 3x) / 9 + (\sin 5x) / 25 - ...]. \tag{14.4}$$

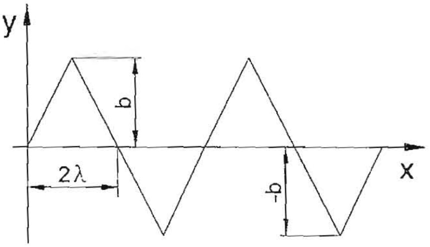

Fig. 14.2. The triangular periodic function.

Practically, any kind of periodic functions can be mathematically expressed by similar equations with the equations (14.3) and (14.4). We know that these equations are enough complicated since for establishing the algebraic terms and factors (non trigonometric) which they comprise is very elaborate. On the other side, these respective equations remain essentially expressed with a certain approximation, their degree of precision depending on the number of the trigonometric terms in the series.

In what follows we will present a simpler method to express mathematically the periodic functions, where we will not use the Fourier series.

14.2. Periodic Functions expressed by using "Matrix Functions", "Matrix Transported Function" and of the "Transport Function"

We consider a function of which the graphical representation in the interval $0 \le x \le 2\lambda$ is as given in Figure 14.3 (bold noted curve).

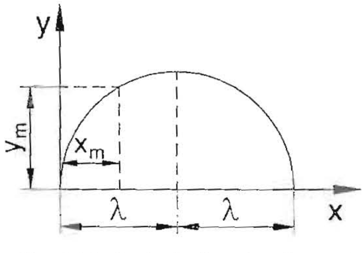

Fig. 14.3. Periodic function formed from the successions of the semi-circles.

We name this "The Matrix Function" (y_M) since this is the one which we intend to transform in a "Periodic Function" (y_P) having the period 4λ. Each semi period of the size 2λ is corresponding to the positive values and respectively to the negative values of the considered function. Thus we have:

$$y_m = y_m(x_m) \tag{14.5}$$

where the variable x_m is in the interval $0 \le x \le 2\lambda$.

To transform the function y_m in a periodic function (y_p) having the period 4λ, thus to extend the function y_m by the periodicity [21], we will "transport" the function y_m along the Ox axis in the entire domain up to $x = +\infty$ and we will do this in a such way that alternatively, at equal intervals with the semi period 2λ, the function will change its algebraic sign. In order to do that we use the trigonometric function "sinus" and we will transform the linear measure of the abscissas x into an angular measure. As a consequence, to the semi-period 2λ it will correspond the semi period π. To the variable with linear character x will correspond the variable with angular character υ and the connection between them is the following:

$$\upsilon = (\pi / 2\lambda) \cdot x \tag{14.6}$$

In order to "transport" the matrix function and to change alternatively its sign at the intervals equal with 2λ, as we have shown above, we introduce the following function expressed by the trigonometric function "sinus":

$$y_t = (\sin \upsilon) / |\sin \upsilon| . \tag{14.7}$$

We name the function y_t the "transport function". If we consider relation (14.6) we have:

$$y_t = [\sin (\pi / 2\lambda) \cdot x] / |\sin (\pi / 2\lambda)| . \tag{14.8}$$

The function y_t have all the time the value equal with 1 (one), but its algebraic sign changes at equal intervals with the semi period π, referred to the variable υ, respectively at the intervals equal with 2λ, if we refer to the variable x – see relation (14.8).

We remember that initially the matrix function y_m is expressed on an interval equal with 2λ relatively to the abscissas axis Ox. For this reason we refer the current coordinate to this interval. Thus we can write $x = 2\lambda \cdot n + x_m$, where n is positive integer number belonging to the natural number system, thus $n = 0, 1, 2, \ldots, +\infty$. In this way in order to use the function y_t (x_m), we will arrive to its current value x "moving" this function along Ox axis "steps" equal with 2λ. Considering all these facts it appears "The transported matrix function" which we denote with y_{mt}. In fact, this is the function y_m where instead of x_m we introduce ($x - 2\lambda n$).

It for example, we have $x = 3 \cdot 3\lambda$, we will write this under the form $x = 2 \cdot \lambda + 1.3 \cdot \lambda$ and we have in this case $n = 1$ and in the matrix function the corresponding abscissa is $x_m = 1.3 \cdot \lambda$.

The desired periodic function which we finally want to have is

$$y_p = y_{mt} \cdot y_t . \tag{14.9}$$

Introducing y_t of relation (14.8) we have

$$y_p = y_{mt} \cdot [\sin (\pi / 2\lambda) \cdot x] / |\sin (\pi / 2\lambda) \cdot x| . \tag{14.10}$$

If we compare the method in the development of periodic functions in Fourier series with the method which we proposed, we observe that in the first case it is done by "composition", in the direction of the Oy axis of a specific periodic function (or of a specific function which is desirable to be periodic), by adding some trigonometric functions, while in our case we proceed in moving in the direction of Ox axis with well established steps, of a know function on an finite interval of the variable x.

In our proposed method case in this paper we eliminate all the disadvantages of the development in Fourier series, mentioned in Subchapter 14.1, in the sense that we have less unnecessary laborious computations and the precision in modeling mathematically this considered function is of 100%.

14.3. Examples of obtained periodic functions starting from Matrix Functions

14.3.1. The case of the line segments parallel with the Ox axis

As a first application example of our method developed in the pervious Subchapter is just the case mentioned in Subchapter 14.1, when we have the Matrix Function $y_{m1} = b$ defined on the interval $0 \leq x \leq 2\lambda$. As a result of our above discussion, since the function y_{m1} is independent of x we will also have the function y_{mt1} independent of x, thus

$$y_{mt1} = b. \tag{14.11}$$

The transport function y_t is given by the relation (14.8) and thus the periodic function y_{p1} will be:

$$y_{p1} = b \cdot [\sin(\pi x / 2\lambda)] / |\sin(\pi x / 2\lambda)|. \tag{14.12}$$

Thus for example, for $x = 3.3 \cdot \lambda$ we will have $y_{p1} = -b$, and for $x = 4.1 \cdot \lambda$ we will have $y_{p1} = +b$.

14.3.2. The case when the Matrix Function is represented by a semi-circle

In Figure 14.3 in fact we represented such a function. The equation which represents the semi-circle Matrix Function, valid in the interval $0 \leq x \leq 2\lambda$ is:

$$(y_{m2})^2 = 2\lambda \cdot x_m - (x_m)^2. \tag{14.13}$$

respectively:

$$y_{m2} = (2\lambda \cdot x_m - x_m{}^2)^{1/2}. \tag{14.14}$$

Starting with the form (14.14) of the equation and applying the method developed in this paper we obtain the transport Matrix Function y_{mt2} replacing in the relation (14.14) the x_m with $(x-2\lambda\cdot n)$, as we have shown in the previous Subchapter. We remember that n is "the integer part of the quotient in dividing x" by 2λ. The periodic function which represents the semi circle succession of Figure 14.3, using the relation (14.8) will be

$$y_{p2} = y_{mt2}\cdot[\sin(\pi x/2\lambda)/|\sin(\pi x/2\lambda)|]. \qquad (14.15)$$

For example, if we accept $x = 2.7\cdot\lambda$ we have $x/2\lambda = 1.35$ and thus $n = 1$. In consequence, replacing in the relation (14.13) x_m with $x-2n\lambda = 2.7\lambda - 2\lambda = 0.7\lambda$, we obtain $y_{mt2} = 0.954\cdot\lambda$. On the other side, the transport function will have the negative value $y_{t2} = -1$ and thus y_{p2} have also a negative value. If we admit $x = 5.1\cdot\lambda$, performing the necessary calculations we have y_{mt2} and $y_{t2} = +1$ and thus the periodic function y_{p2} will have a positive value.

14.4. Conclusions of the Chapter 14

In presenting the previous Subchapters we have the following more important conclusions:

14.4.1. The Fourier Method in developing periodic functions in trigonometric series is laborious and does not have a 100 % precision. In order to increase precision there is a need to increase the number of the terms of the respective trigonometric series.

14.4.2. The mathematical modeling Method of a periodic function presented in this Chapter starts from a basic function valid for a limited domain of its variable. We named this function the "Matrix Function" and we proceed to extend its periodicity.

For this reason we introduced two types of functions which were named the "Transported Matrix Function" and respectively, the "Transport Function".

14.4.3. The Method developed in this paper is much simpler then the Fourier Method and ensures a degree of precision in its mathematical modeling of 100%. This fact was illustrated by examples referring to two types of Matrix Functions.

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