THE ALGORITHMIC SOLUTION OF THE ORIGINAL EUCLIDEAN FERMAT'S LAST THEOREM (EFLT)

Chapter I. KNOWN ALGORITHMS FOR REAL NUMBERS

Section 1.0. Introduction

A Wiles [46] anounced his proof of the eliptical Fermat's Last Theorem (ELFLT) in the Summer of 1993.

At that time the author was concerned that this (ELFLT) is not the same as Fermat's Last Theorem in Euclidean terms (EFLT) where Fermat's Last Theorem originated about three hundred fifty ysears ago. Also, the author was concerned about using a low level when the higher levels of the Euler System were not constructed in the number theory of the elliptic curves.

Hasse, who was the author's Ph.D. dissertation advisor, once stated: „The end of the 20^{-th} century will bring the solution for Fermat's Last Theorem (FLT) and the solution will come from the Euclidean Number Theory tools, as Fermat had intented". Gauss invented the Algebraic Number Theory to be the Algebra of the n-Dimensional Euclidean Geometry (E^n G) and therefore it has to be proved in the Algebraic Number Theory.

In May 1994, the author presented her Algorithmic Euclidean proof of (EFLT) [13] at the International Conference on Analytic Number Theory in Allerton Park, at the University of Illinois in Urbana – Champaign.

A Wiles's proof was declared finally completed on September of 1994.

In 1995, The Annals of Mathematics, a publication at Princeton [44, 46] accepted a proof of (FLT) in the geomety of elliptic curves and its corresponding number theory which is Hecke Algebra.

Gauss, who is called The Prince of Mathematicians, said that he was not going to waste his proessional life in attemting to prove (FLT) wich he may not have been able to prove anyway, and that he would leave the problem to better mathematicians.

The July 1995, Notices of the AMS published a translation from the German of G. Faltings' March 1995 article [32] in which the author said at the bigining of the paper: „The proof of the contecture mentioned in the title was finally completed in Sptember of 1994. A Wiles announced this result in the Summer of 1993. However, there was a gap in his work. "The paper of Taylor and Wiles [44] does not close this gap but circumvents it".

With this statement one of my concerns was justified. A Wiles did not prove (ELFLT). It is G. Faltings who did it. In this book we will show that Faltings' (ELFLT) is equivalent to the original (EFLT) proved by Baica using her Generalized Euclidean Algorithm (BGEA). With this we show to the Annals of Mathematics that Baica's Generalized Euclidean Algorithm (BGEA) has everything to do with the Euclidean solution of (EFLT).

(ELFLT) is equivalent to (not the same as) with the original Fermat's Last Theorem (EFLT) stated by Fermat in Euclidean Terms.

Section 1.1. The Euclidean Algorithm (EA), continued fraction algorithm and Euler-Lagrange Theorem (2 - ELT) for quadratics.

The basis of most of the results in this book is an algorithm, and we shall therefore give a short historical survey of ist development. It all started with the very well known Euclidean Algorithm (EA) which is an iteration of the division algorithm, known to Euclid more than 2000 years ago. For instance using the Euclidean Algorithm, it is easy to prove that every rational number can be represented as a finite simple continued fraction. For that we will define the Division Algorithm (DA) of positive integers and use it to give the simple continued fraction interpretation of the (EA).

Let consider the Division Algorithm (DA) of positive integers. A_0 and A_1 with $A_0 > A_1 > 0$, then there exists uniquely determined integers A_2 and b_0

such that

$$A_0 = A_1 b_0 + A_2 \text{ where } A_1 \neq 0 \text{ and } 0 \leq A_2 < A_1 \text{ with } b_0 = \left[\frac{A_0}{A_1}\right].$$

Iteration of the (DA) leads to the greatest common divisor of A_0 and A_1 denoted by $g\,c\,d\,(A_0,\ A_1)$ and it is known as the Euclidean Algorithm (EA).

Another interpretation of the (EA) is the Simple Continued Fraction Algorithm. We put

$$a_0 = \frac{A_0}{A_1}\ ,\quad a_1 = \frac{A_1}{A_2} \qquad (A_2 > 0)$$

then $a_0, a_1 > 1$,

$$a_0 = b_0 + \frac{1}{a_1}\ ,\ \text{where } b_0 = [a_0]$$

$$a_1 = b_1 + \frac{1}{a_2}\ ,\ \text{where } b_1 = [a_1]$$

....

$$a_n = b_n \ (\neq 1) \ \text{ where } b_n = [a_n]$$

The Algorithm terminates if a_n becomes an integer, necessarily greater than 1. In this way we have a one correspondence between rationals $a_0 > 1$ and these finite sequences $\{b_0, b_1, \ldots, b_n\}$ of positive integers $b_0, b_1, \ldots, b_n$ so that $b_n \neq 1$.

a_0 can be writen as a afine continued fraction sequence

$$\{b_0, b_1, \ldots, b_n\}.$$

$$a_0 = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}$$

Thus we proved that using (EA) every rational number can be represented as a finite simple continued fraction.

Example 1.1.1.

$$\left[\frac{534}{27}\right] = 19 + \frac{21}{27} = 19 + \frac{1}{\frac{27}{21}} = 19 + \frac{1}{1+\frac{6}{21}} =$$

$$= 19 + \frac{1}{1+\frac{1}{\frac{21}{6}}} \;=\; 19 + \frac{1}{1+\frac{1}{3+\frac{3}{6}}} =$$

$$= \; 19 + \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{\frac{6}{3}=2}}} \;=\; 19 + \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2}}}$$

therefore

$$\frac{534}{27} = \{19, 1, 3, 2\}.$$

Let the starting vector be $a^{(0)} = \left(a_1^{(0)}\right) \in \mathbb{R}$ and a transformation function which is the greatest integer function $\left[a_1^{(0)}\right]$ as a companion vector

$$b^{(0)} = \left[a_1^{(0)}\right] = \left(b_1^{(0)}\right) \in \mathbb{R},$$

then the recursive transformation

$$a^{(v+1)} = \left(a_1^{(v)} - b_1^{(v)}\right)^{-1}$$

applied to this vectors becomes a sequence $\{a^{(v)}\}$, $v = 0, 1, \ldots$, which is called the continued fraction interpretation of (EA).

Now use this interpretation to find the finite sequence of

$$\frac{534}{27} = a^{(0)}.$$

Example 1.1.2.

$$a^{(0)} = \left(a_1^{(0)}\right) = \frac{534}{27} \qquad b^{(0)} = \left(b_1^{(0)}\right) = \left[\frac{534}{27}\right] = 19 = b_1^{(0)}$$

$$a^{(1)} = \frac{1}{a_1^{(0)} - b_1^{(0)}} = \frac{1}{\frac{534}{27} - 19} = \frac{1}{\frac{21}{27}} = \frac{27}{21} = a_1^{(1)}\,;\ b_1^{(1)} = 1$$

$$a^{(2)} = \frac{1}{a_1^{(1)} - b_1^{(1)}} = \frac{1}{\frac{27}{21} - 1} = \frac{1}{\frac{6}{21}} = \frac{21}{6} = a_1^{(2)}\,;\ b_1^{(2)} = 3$$

$$a^{(3)} = \frac{1}{a_1^{(2)} - b_1^{(2)}} = \frac{1}{\frac{21}{6} - 3} = \frac{1}{\frac{3}{6}} = \frac{6}{3} = a_1^{(3)}\,;\ b_1^{(3)} = 2 \quad \text{positive integer}$$

$$\frac{534}{27} = \{b_1^{(0)},\, b_1^{(1)},\, b_1^{(2)},\, b_1^{(3)}\} = \{19, 1, 3, 2\}.$$

This formula

$$(1.1.1.) \qquad a^{(v+1)} = \frac{1}{a_1^{(v)} - b_1^{(v)}}$$

defines the continued fraction algorithm which coincide with the Euclidean Algorithm (EA).

Lagrange (1770) and Euler (1737) proved that every real quadratic irrational is represented by a periodic infinite simple continued fraction sequence and that every periodic infinite simple continued fraction sequence represents a real quadratic irrational. This theorem is known as Lagrange`s Theorem in Number Theory and is started as follows:

Euler – Lagrange Theorem 1.1.1.

We have a one to one correspondence between irrationals $\alpha_0 > 1$ and infinite sequences (Simple continued fractions sequences) $\{b_1, b_2, \ldots, b_n, \ldots\}$.

The sequence is periodic if and only if α_0 is algebraic of degree 2 (root of an irreducible rational polynomial of degree 2).

In 1737, Euler proved that periodicity implies α_0 is algebraic and in 1770, Lagrange proved the converse. Based on these historical facts we refer to Lagrange's Theorem as the Euler-Lagrange Theorem for Quadratics (2 – ELT).

(2 – ELT) proves that every quadratic irrational makes (EA) periodic and it proves that the original (EA) is always periodic.

Example 1.1.3.

This is the simple continued fraction development for $a^{(0)} = \left(\sqrt{3}\right)$

$$a^{(0)} = \left(\sqrt{3}\right);\ a_1^{(0)} = \sqrt{3};\quad b_1^{(0)} = 1;\quad b^{(0)} = (1)$$

$$a^{(1)} = \frac{1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2} = a_1^{(1)};\ b_1^{(1)} = 1;\ b^{(1)} = (1)$$

$$a^{(2)} = \frac{1}{\frac{\sqrt{3}+1}{2}-1} = \frac{2}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} = \sqrt{3}+1 = a_1^{(2)};\ b_1^{(2)} = 2;\ b^{(2)} = (2)$$

$$a^{(3)} = \frac{1}{\sqrt{3}+1-2} = \frac{1}{\sqrt{3}-1} = \frac{1}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2} = a^{(1)}$$

$$\sqrt{3} = \left[b^{(0)},\ \overline{b^{(1)}, b^{(2)}}\right] = \left[1, \overline{1, 2}\right]$$

Example 1.1.4.

Likewise for $a^{(0)} = \left(\sqrt{7}\right)$

$$a^{(0)} = \left(\sqrt{7}\right);\ a_1^{(0)} = \sqrt{7};\ b_1^{(0)} = 2;\ b^{(0)} = (2)$$

$$a^{(1)} = \frac{1}{\sqrt{7}-2}\cdot\frac{\sqrt{7}+2}{\sqrt{7}+2} = \frac{\sqrt{7}+2}{3}\quad;\ b^{(1)} = (1)$$

$$a^{(2)} = \frac{1}{\frac{\sqrt{7}+2}{3}-1} = \frac{3}{\sqrt{7}-1}\cdot\frac{\sqrt{7}+1}{\sqrt{7}+1} = \frac{\sqrt{7}+1}{2};\ b^{(2)} = (1)$$

$$a^{(3)} = \frac{1}{\frac{\sqrt{7}+1}{2}-1} = \frac{2}{\sqrt{7}-1}\cdot\frac{\sqrt{7}+1}{\sqrt{7}+1} = \frac{\sqrt{7}+1}{3};\ b^{(3)} = (1)$$

$$a^{(4)} = \sqrt{7}+2;\ b(4) = (4)$$

$$a^{(5)} = \frac{1}{\sqrt{7}+2-4}\cdot\frac{\sqrt{7}+2}{\sqrt{7}+2} = \frac{\sqrt{7}+2}{3} = a^{(1)}\quad b^{(5)} = (1)$$

thus

$$a^{(0)} = \sqrt{7} = \left[2,\ \overline{1,\ 1,\ 1,\ 4}\right].$$

Section 1.2. The Euclidean Algorithm is the Euler System in Quadratics.

Since we cannot compute in a geometry , it is known that to every geometry we can associate a corresponding algebra, although the converse is not true, and we call this Algebra the Number Theory (the arithmetic) corresponding to that geometry.

In 1940, E. Schmidt was the first to say that because we cannot compute in a geometry, we have to construct an algebra for that corresponding geometry. He is the father of the General Algebraic Geometry as C.F.Gauss is the father of the Algebraic Number Theory, which is the Algebra of the n-Dimensional Euclidean Geomertry (E^nG). In every Number Theory there is a very strong theorem where if we implement rightly the conditions of other new theorems in the conitions of this main strong theorem, then these new theorems become immediate consequences of this initial strong theorem.

This is known in modern mathematics as the Euler System (ES) of that number theory. In other words, the (ES) in that corresponding number theory is a very powerful tool to prove many theorems in that Number Theory.

The unrestricted periodicity of the (EA) (that is that <u>every</u> quadratic irrational makes the (EA) periodic) is a very strong property, and because of this many important results in the Algebraic Number Theory in Quadratics were completly solved from its periodicity. The same problems remain open in n-Dimensions, because there did not exist an unrestricted periodic generalized Euclidean Algorithm (GEA) to solve them from its periodicity.

Some previous results from the unrestricted periodicity are:

(1.2.1) Construction with the ruler and the compass of the quadratic irrationals on the real line.

(1.2.2.) Every real quadratic irrational can be represented by an infinite periodic continued fraction (EA) development. This is (2 - ELT).

(1.2.3.) Explicit solution of the Euler–Pellian equations $x^2 - ay^2 = \pm 1$ and ± 4.

(1.2.4) The problem of finding the multiplicative group of units in the quadratic algebraic number fields was completely solved when Euler-Pellian's equations in (1.2.3) were completely solved. This is Dirichlet's problem for $n = 2$.

(1.2.5) The existence of an algorithm to approximate quadratic irrationals.

(1.2.6) The existence of the quadratic equation formula.

(1.2.7) The determination of Pythagorean triples which leads to the integer solutions of $x^2 + y^2 = z^2$.

The corresponding question for $n > 2$ were unsolved because there was no (GEA) to prove them, and this caused Hilbert to ask for the invention of a universal algorithm as powerful as (EA) for $n = 2$ in order to solve all of the n-Dimensional open questions for $n > 2$ from the periodicity of this universal algorithm, just as the above mentioned problems were solved form the unrrestricted periodicity of the (EA). This is known as Hilbert's 10-th Problem.

In [23] the author showed that because of this, (EA) is the (ES) in the Number Theory of Quadratics (E^2 G).

Section 1.3. Jacobi Algorithm (JA).

Mathematicians had almost abandoned hope of obtaining further information about the arithmetic properties of higher degree algebraic irrationals by means of a simple continued fraction (or EA), when Jacobi [36] generalized the Euclidean Algorithm for the cubic case.

In 1839, Hermite [34] in one of his letters to Jacobi, challenged Jacobi to find an algorithm to develop irrationals of any degree into periodic sequences. Hermite was asking for the general simple periodic continued fractions algorithm. But it was only after thirty years of frustration that Jacobi in 1868 extended (EA) methodes to successfully represent some cubic irrationals by means of simple continued fractions.

An application of the (JA) starts with the initial vector

$$a^{(0)} = \left(a_1^{(0)}, a_2^{(0)}\right) \in \mathbb{R}^2,$$

$n = 3$, the components of which are algebraic numbers. By use of the greatest integer function a „companion vector"

$$b^{(0)} = \left(b_1^{(0)}, b_2^{(0)}\right) \in \mathbb{R}^2,$$

with $b_i^{(0)} = \left[a_i^{(0)}\right]$, $i = 1, 2$ is defined. A recursive transformation

$$a^{(v+1)} = \left(a_1^{(v)} - b_1^{(v)}\right)^{-1}\left(a_2^{(v)} - b_2^{(v)}, 1\right)$$

is constructed and applied to these vectors. Then the sequence $\{a^{(v)}\}$, $v = 0, 1, 2\ldots$; is called Jacobi Algorithm (JA) of $a^{(0)}$.

Example 1.3.1.

$$a^{(0)} = \left(a_1^{(0)}, a_2^{(0)}\right) = \left(\sqrt[3]{2}, \sqrt[3]{4}\right) \qquad b^{(0)} = \left(b_1^{(0)}, b_2^{(0)}\right) = (1, 1)$$

$$a^{(1)} = \left(\sqrt[3]{2} - 1\right)^{-1}\left(\sqrt[3]{4} - 1, 1\right)$$

$$a^{(1)} = \left(\frac{\sqrt[3]{4} - 1}{\sqrt[3]{2} - 1}, \frac{1}{\sqrt[3]{2} - 1}\right) = \left(a_1^{(1)}, a_2^{(1)}\right)$$

$$a_1^{(1)} = \frac{\left(\sqrt[3]{4} - 1\right)\cdot\left(\sqrt[3]{4} + \sqrt[3]{2} + 1\right)}{\left(\sqrt[3]{2} - 1\right)\cdot\left(\sqrt[3]{4} + \sqrt[3]{2} + 1\right)} = \frac{2\sqrt[3]{2} - \sqrt[3]{4} + 2 - \sqrt[3]{2} + \sqrt[3]{4} - 1}{2 - 1} =$$

$$= \sqrt[3]{2} + 1$$

$$a_2^{(1)} = \frac{1\cdot\left(\sqrt[3]{4} + \sqrt[3]{2} + 1\right)}{\left(\sqrt[3]{2} - 1\right)\cdot\left(\sqrt[3]{4} + \sqrt[3]{2} + 1\right)} = \frac{\sqrt[3]{4} + \sqrt[3]{2} + 1}{2 - 1} = \sqrt[3]{4} + \sqrt[3]{2} + 1$$

$$a^{(1)} = \left(\sqrt[3]{2} + 1, \sqrt[3]{4} + \sqrt[3]{2} + 1\right), \quad b^{(1)} = (2,3)$$

$$a^{(2)} = \left(\frac{\sqrt[3]{4} + \sqrt[3]{2} + 1 - 3}{\sqrt[3]{2} - 1}, \frac{1}{\sqrt[3]{2} - 1}\right) = \left(a_1^{(2)}, a_2^{(2)}\right)$$

$$a_1^{(2)} = \frac{\left(\sqrt[3]{4} + \sqrt[3]{2} - 2\right)\cdot\left(\sqrt[3]{4} + \sqrt[3]{2} + 1\right)}{\left(\sqrt[3]{2} - 1\right)\cdot\left(\sqrt[3]{4} + \sqrt[3]{2} + 1\right)} = \sqrt[3]{2} + 2$$

$$a_2^{(2)} = \frac{1\cdot\left(\sqrt[3]{4}+\sqrt[3]{2}+1\right)}{\left(\sqrt[3]{2}-1\right)\cdot\left(\sqrt[3]{4}+\sqrt[3]{2}+1\right)} = \frac{\sqrt[3]{4}+\sqrt[3]{2}+1}{2-1}$$

$$a^{(2)} = \left(\sqrt[3]{2}+2,\ \sqrt[3]{4}+\sqrt[3]{2}+1\right),\quad b^{(2)} = (3,3)$$

$$a^{(3)} = \left(\frac{\sqrt[3]{4}+\sqrt[3]{2}+1-3}{\sqrt[3]{2}+2-3},\ \frac{1}{\sqrt[3]{2}-1}\right) = \left(a_1^{(3)},\ a_2^{(3)}\right)$$

$$a_1^{(3)} = \frac{\left(\sqrt[3]{4}+\sqrt[3]{2}-2\right)\cdot\left(\sqrt[3]{4}+\sqrt[3]{2}+1\right)}{\left(\sqrt[3]{2}-1\right)\cdot\left(\sqrt[3]{4}+\sqrt[3]{2}+1\right)} = \sqrt[3]{2}+2 = a_1^{(2)}$$

$$a_2^{(3)} = \frac{1\cdot\left(\sqrt[3]{4}+\sqrt[3]{2}+1\right)}{\left(\sqrt[3]{2}-1\right)\cdot\left(\sqrt[3]{4}+\sqrt[3]{2}+1\right)} = \frac{\sqrt[3]{4}+\sqrt[3]{2}+1}{2-1} = a_2^{(2)}$$

$$a^{(3)} = \left(\sqrt[3]{2}+2,\ \sqrt[3]{4}+\sqrt[3]{2}+1\right) = a^{(2)},\quad b^{(3)} = (3,3) = b^{(2)}.$$

(JA) of $\sqrt[3]{2}$ and $\sqrt[3]{4}$ is periodic and (JA) of

$$\left(\sqrt[3]{2},\ \sqrt[3]{4}\right) = \{(1,1),\ (2,3),\ \overline{(3,3)}\}$$

or

$$\sqrt[3]{2} = \{1,\ 2, \overline{3}\} \text{ and } \sqrt[3]{4} = \{1, \overline{3}\}.$$

For good choices of the starting vector $a^{(0)}$ and transformation $b^{(0)}$, the iteration of the transformation becomes periodic, that is the transformation cycles around a finite set of vectors. In this instance (JA) is said to be periodic and the results lead to the (JA) periodic representation of third degree irrationals. The difficulties associated with this work are many. Jacobi's results were confined to a few examples in a cubic field, where Jacobi exhibited periodic developments for $\sqrt[3]{2}$, $\sqrt[3]{4}$, $\sqrt[3]{3}$, $\sqrt[3]{9}$, $\sqrt[3]{5}$, $\sqrt[3]{25}$.

This problem is known as Hermite's problem for higher degree irrationals. In spite of all Jacobi's efforts Hermite's problem remain unsolved.

Advances were slow and difficult, but in 1873 Bachman proved results for other cubic irrationals using the (JA), results that were accompanied by many restrictions.

Section 1.4. Perron Algorithm (PA).

In 1907, Perron [40] generalized the work of Jacobi. This generalization is known as the Jacobi-Perron Algorithm (JPA).

In its general form, as defined by Jacobi for $n = 3$ and by Perron for $n \geq 2$, an application of the (JPA) starts with the definition of an initial vector

$$a^{(0)} = \left(a_1^{(0)},\, a_2^{(0)}, \ldots, a_{n-1}^{(0)}\right) \in \mathbb{R}^{n-1},\ n \geq 2,$$

the components of which are algebraic numbers. By use of the gratest integer function a "companion vector"

$$b^{(0)} = \left(b_1^{(0)},\, b_2^{(0)}, \ldots, b_{n-1}^{(0)}\right) \in \mathbb{R}^{n-1}, \text{ with}$$

$$b_i^{(0)} = \left[a_i^{(0)}\right],\ (i = 1, 2, \ldots, n\text{-}1)$$

is defined. A recursive transformation

$$a^{(v+1)} = \left(a_1^{(v)} - b_1^{(v)}\right)^{-1} \left(a_2^{(v)} - b_2^{(v)}, \ldots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1\right) \tag{1.4.1}$$

is constructed and applied to these vectors. Then the sequence

$\{a^{(v)}\},\ v = 0, 1, 2 \ldots$; is called (JPA).

Perron generalized Jacobi's methods to irrationals of any degree but since the choices of startindg vector and transformation are difficult to make, he was also limited to a few periodic developments of higher degree irrationals. These results were to prove periodicity for higher degree irrationals, also. With all Perron's efforts periodicity in proving Hermite's problem remains open.

Perron was more successful in showing that if a development is periodic then the components of the initial vector are algebraic numbers. This later results was general.

With this work on Hermite's problem progress come to a halt, because of the failure of the (JPA) to produce new numerical results, that is, additional cases in which the transformation become periodic were not achieved.

Perron and all others recognised that the usual choices for starting vector were too limited. No further progress occurred on these problems until Hasse and Bernstein [24] turned their attention to them.

Section 1.5. Hasse and Bernstein Algorithm (HBA).

In 1965, Hasse and Bernstein made a broader approach to the periodicity problem associated with the (JPA). Hasse and Bernstein started with an algebraic extension of the rational numbers, Q(w), where w takes the form $w = \sqrt[n]{D^n + d}$ with

$$P(x) = \left(\prod_{i=1}^{n} (x^n - D_i^n) - d \right), \quad d \in \mathbb{Z}, \ D_i \in \mathbb{N} \text{ and } d/D.$$

$$a^{(0)} = ((w - D_1)\cdot(w - D_2)\cdot \ldots \cdot(w - D_{n-1}), \ldots, (w - D_1)\cdot(w - D_2), (w - D_2))$$

with $b^{(0)} = a^{(0)}(D_1)$.

They showed that certain significant restrictions on D and d led to a (JPA) that was purely priodic (that is that the lengh of the preperiod is zero).

1) For $d > 0$ they proved that the (JPA) of $a^{(0)}$ is purely periodic when

$$D \geq (n-2)\cdot d, \ d/D \text{ and } n \geq 3, \text{ and}$$

2) For $d < 0$ the sequence is also purely periodic when $D \geq 2\,(n\text{-}1)\,d$, d/D and $n \geq 3$.

With these conditions, the length of the period is $n\cdot(n\text{-}1)$. For this approach the periodicity remains an open problem since there are bounds on D and the restriction d/D must hold. For example no periodicity for $w = \sqrt[5]{12^5 + 6}$ can be proved under (HBA) restrictions since $12 \geq (5-2)\cdot 6 = 18$ is not true.

The Hasse and Bernstein results were limited by their choices of w as real numbers. It should be known that Hasse and Bernstein were not interested in Hermite's problem in spite of the fact that their results had a strong relation to that problem. Specifically, they did not realise that the periodicity of their algorithm leads to a solution of Hermite's problem for some real algebraic number w, when (HBA) becomes the general continued fractions algorithm. There are more n-degree irrationals which have periodic (HBA) algorithmic development than have a general periodic continued fraction development or a periodic (JPA) algorithmic development. Hasse and Bernstein were interested in solving Dirichlet's problem to find units in algebraic number fields from the periodicity of their algorithm. From these results they proved that in both cases (1) and (2)

$$e^k = \frac{w^k - D^k}{(w-D)^k}\ ,\ k/n\,, k > 1$$

are the $\tau(n)$-1 units in the corresponding fields

$$Q(w), w = \sqrt[n]{D^n + d}\ ,\ d/D,\ D \in \mathbb{N}, d \in \mathbb{Z},\ n \geq 3.$$

The shortcomings of this very important result are the restriction on d and the bounds on D.

As of this result the periodicity of their algorithm is an open question, too.