Chapter II. BAICA'S GENERALIZED EUCLIDEAN ALGORITHM (BGEA)

Section 2.0. Introdunction

In 1980, the author [1] defined a midification of the (JPA) that used the Hasse and Bernstein initial vector, but was not restricted to the real numbers. For the first time the complex numbers were considered. The only differences in the definitions stated alone are that the D_i's are now complex numbers. An immediate consequence of this extensions is that the bounds on D in the (HBA) are now eliminated and only the divisibility condition, *d*/D, remains.

Returning to the example cited in the Section 1.5. it can now be seen that $w = \sqrt[5]{12^5 + 6}$ has a periodic development, only 6/12 is required. At that time Baica proved only that *d*/D is necessary condition to make her algorithm to be periodic and named her algorithm, the algorithm for complex field (ACF). Later when Baica proved that *d*/D is also a sufficient condition for the periodicity of her (ACF) algorithm then (ACF) becames Baica's General Euclidean Algorithm (BGEA).

In this book we refer to Baica's Generalized Euclidean Algorithm as (BGEA).

Section 2.1. Baica's Generalized Euclidean Algorithm (BGEA); a different proof for the necessary condition of the restricted periodicity.

In this section we will give a different proof from [1] for the necessary condition of the restricted periodicity.

Definition 2.1.1. Let w be a real root of an irreductible polynomial of degree $n \geq 2$ over Q. A sequence of $(n\text{-}1)$-dimensional vectors $a^{(v)}$ will be called (BGEA) of the initial vector.

$$(2.1.1) \quad a^{(0)} = \left(a_1^{(0)}(w),\, a_2^{(0)}(w),\, \ldots,\, a_{n-1}^{(0)}(w)\right),$$

$$a_i^{(0)}(w) \in \mathbb{C}[w], \quad (i = 1, 2, \ldots, n\text{-}1)$$

if

$$(2.1.2) \begin{cases} a^{(v+1)} = \left(\left(a_1^{(v)} - b_1^{(v)}\right)^{-1}\left(a_2^{(v)} - b_2^{(v)}, \ldots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1\right)\right) \\ (v = 0, 1, \ldots) \textit{ where for } i = 1, \ldots, n-1,\ a_i^{(v)} = a_i^{(v)}(w) \in C \textit{ and} \\ b_i^{(v)} = a_i^{(v)}\left(\rho^t D\right), \textit{ for a fixed integer } D \textit{ with} \\ \rho = e^{\frac{2\pi i}{n}}, \textit{ and } t \textit{ is any fixed integer of } \{0, 1, \ldots, n-1\}. \end{cases}$$

Following Bernstein [25], we now introduce complex numbers $A_i^{(v)}$ as follows.

$$(2.1.3) \begin{cases} A_i^{(j)} = \delta_{ij},\ 0 \le i,\ j \le n-1 \quad (\delta_{ij} \textit{ being Kronecker symbol}) \\ A_i^{(n+v)} = A_i^{(v)} + \sum\limits_{k=1}^{n-1} b_k^{(v)}\, A_i^{(k+v)};\ v = 0, 1, \ldots \end{cases}$$

Thus for instance, we have

$$A_0^{(0)} = 1,\ A_0^{(1)} = A_0^{(2)} = \ldots = A_0^{(n-1)} = 0;$$

$$A_0^{(n)} = A_0^{(0)} + \sum_{k=1}^{n-1} b_k^{(0)}\, A_0^{(k)} = A_0^{(0)} + 0 = 1$$

$$A_0^{(n+1)} = A_0^{(1)} + \sum_{k=1}^{n-1} b_k^{(1)}\, A_0^{(k+1)} = b_{n-1}^{(1)} + A_0^{(n)} = b_{n-1}^{(1)}$$

$$A_1^{(0)} = 0,\ A_1^{(1)} = 1,\ A_1^{(2)} = A_1^{(3)} = \ldots = A_1^{n-1} = 0.$$

Bernstein [25] has proved that for any algorithm

$$a^{(v+1)} = \left(a_1^{(v)} - b_1^{(v)}\right)^{-1}\left(a_2^{(v)} - b_2^{(v)}, \ldots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1\right)$$

with vectors $a^{(v)} \in \mathbb{R}^{n-1}$ and a fixed vector $a^{(0)}$, where

$$b^{(v)} = \left(b_1^{(v)}, \ldots, b_{n-1}^{(v)}\right)$$

is any given sequence of vectors, the following formulas hold, if the numbers $A_i^{(v)}$ are defined by (2.1.3).

$$(2.1.4) \qquad \begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ \vdots & \vdots & & \vdots \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = (\text{-}1)^{v(n\text{-}1)}, \ (v = 0, 1, \ldots);$$

$$(2.1.5) \qquad a_i^{(0)} = \left[A_i^{(v)} + \sum_{k=1}^{n-1} a_k^{(v)} A_i^{(k+v)} \right] \Big/ \left[A_0^{(v)} + \sum_{k=1}^{n-1} a_k^{(v)} A_0^{(k+v)} \right];$$

$$(v = 0, 1, \ldots);$$

$$(2.1.6) \qquad \prod_{k=1}^{v} a_{n-1}^{(k)} = A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(j+v)} \ ; \ \ (v = 1, 2, ,,,);$$

$$(2.1.7) \qquad \begin{vmatrix} 1 & A_0^{(v+1)} & A_0^{(v+2)} & \cdots & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & A_1^{(v+2)} & \cdots & A_1^{(v+n-1)} \\ a_2^{(0)} & A_2^{(v+1)} & A_2^{(v+1)} & \cdots & A_2^{(v+n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & A_{n-1}^{(v+2)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} =$$

$$= \frac{(-1)^{v(n-1)}}{A_0^{(v)} + \sum_{j=1}^{n-1} a_j(v) \cdot A_0^{(j+v)}} \, .$$

For $v = 0$ the determinant on the left of (2.1.7) simplifies to 1 by (2.1.3) (*i*) and the expression on the right side becomes $(\text{-}1)^0$ / 1 + 0 = 1. Bernstein proved formulas (2.1.4) – (2.1.7) by induction. The most important formulas for our purpous are the last two and we shall apply them next.

The reader should note carefully the structure of $b_i^{(v)} = a_i^{(v)}\left(\rho^t D\right)$, for any fixed integer D, where $0 \le t \le n\text{-}1$. Actually, t could be any natural number, since $\rho^n = 1$. But once a value for t is choosen in a particular (BGEA), it must remain fixed throughout the sequel.

Now, we shall state a (BGEA) whose periodicity is one of the main results of this book. We shall carry out the calculations for $n = 4$ in detail (2.1*). For $n = 5$ (2.2*) we shall write down the respective vectors $a^{(v)}$ and $b^{(v)}$

without making the necessary calculations and from the pattern of the (BGEA) for these two cases we shall give the proof for the general (BGEA) by induction (2.3*).

In [24] Bernstein and Hasse proved similar results for (JPA) with extra conditions determined from the fact that $D_i \in \mathbb{N}$. For $n = 4$ and $n = 5$ the development of (BGEA) is contained in [24] except that the D_i's are complex numbers. The results are repeated or slightly modified with regard to the (BGEA). For the general case n of (BGEA), the author proved the periodicity by induction.

Theorem (2.1.1)

Let $D \in \mathbb{N}$, $d \in \mathbb{Z}$, d/D. Then

$$w = \sqrt[n]{D^n + d}\ , \ n \geq 2 \tag{2.1.8}$$

is an algebraic integer of degree n.

Bernstein [25] proved this theorem for both cases $d > 0$ and $d < 0$, but for completeness we prove this theorem here again. For $d > 0$ we repeat Bernstein's proof in [25] but for $d < 0$ we give a simple algebraic proof instead of Bernstein's analytic proof.

Proof: Let us first prove it for $d > 0$, w is root of

$$x^n - (D^n + d) = 0, \tag{2.1.9}$$

hence it is an algebraic integer, by hypothesis.

To prove that w is of degree n, we have to show that the polynomial

$$P(x) = x^n - (D^n + d) \text{ is irreductible over } Q. \tag{2.1.10}$$

The roots of (2.1.9) are

$$\begin{cases} x_k = \sqrt[n]{D^n + d} \\ w = x_0. \end{cases} \rho_k\ , \ \rho_k = \exp(2\pi k i/n),\ k = 0, 1, \ldots, n-1 \tag{2.1.11}$$

We have $w^n = D^n + d \in \mathbb{N}$.

Let $k(\leq n)$ be the least positive integer such that $w^k \in \mathbb{N}$.

Then $n = q\,k + r$ whence $r = 0$ so k/n.

Now $D^n < D^n + d < D^n + D$ if $d > 0$.

Suppose $k < n$.

Hence $\mathrm{D}^n < \mathrm{D}^n + d < (\mathrm{D}^k + 1)^{n/k}$

This gives $\mathrm{D}^k < (\mathrm{D}^n + d)^{k/n} < \mathrm{D}^k + 1$

or $\mathrm{D}^k < w^k < \mathrm{D}^{k+1}$, the required contradiction to $w^n \in \mathbb{N}$.

Now let $d < 0$.

$$w^n = \mathrm{D}^n - d.$$

We now show $\mathrm{D}^k > w^k > D^k$ -1. Let $n_0 = n / k$.

$$D^{k n_0} > w^{k n_0}$$

$$\mathrm{D}^n > \mathrm{D}^n - \mathrm{D}.$$

Let us show $w^k > \mathrm{D}^k - 1$.

$$w^n = w^{k n_0} > (D^k - 1)^{n_0}.$$

$$\mathrm{D}^n - d = (D^k)^{n_0} - d > (D^k - 1)^{n_0}.$$

$$(D^k)^{n_0} - (D^k - 1)^{n_0} > d.$$

We have $(D^k)^{n_0} - (D^k - 1)^{n_0} = x^{n_0} - (x-1)^{n_0}$ say, with $x = \mathrm{D}^k$.

$$= [x - (x-1)]\left[x_0^{n_0 - 1} + x^{n_0 - 2}(x-1) + \ldots + (x-1)^{n_0 - 1}\right] >$$

$$> n_0 (x-1)^{n_0 - 1} \geq 2(x-1), \quad (n_{\circ} = n / k \text{ where } k / n,\ k < n)$$

$$= 2\,(\mathrm{D}^k - 1) \geq 2\,(\mathrm{D} - 1) \geq \mathrm{D} \geq d.$$

(provided $\mathrm{D} \geq 2$, which is true).

So $w^k > \mathrm{D}^k$ -1 and w^k is irrational. Thus we arrived at a contradiction. Since w is of degree k, then w^k must $\in \mathbb{N}$.

Hence w has exactly the degree n.

2.1*. A periodic (BGEA), *n* = 4

Suppose

$$(2.1^*.1) \qquad w = \sqrt[4]{D^4 + d}\,, \quad D,\ |d| \in \mathbb{N} \quad \mathrm{d} / \mathrm{D}.$$

Now since $w^4 = \mathrm{D}^4 + d$, $(w^4 - \mathrm{D}^4) - d = 0$, we have

$$(2.1^*.2) \qquad (w - \mathrm{D})\,(w - \rho\,\mathrm{D})\,(w - \rho^2\,\mathrm{D})\,(w - \rho^3\,\mathrm{D}) - d = 0$$

where $\rho = \exp\,(\pi\, i / 2)$.

For brevity, write

$$(2.1^*.3) \qquad \{D_1, D_2, D_3, D_4\} \text{ for } \{d, \rho D, \rho^2 D, \rho^3 D\}$$

taken in some order, i.e. $\{D_1, D_2, D_3, D_4\}$ is some permutation of the numbers $\{d, \rho D, \rho^2 D, \rho^3 D\}$. We shall now carry out the (BGEA) with the initial vector

$$(2.1^*.4) \qquad a^{(0)} = ((w - D_1)(w - D_2)(w - D_3), (w - D_1)(w - D_2), (w - D_2)).$$

From (2.1*.2) and (2.1*.3) we obtain

$$(2.1^*.5) \qquad (w - D_1)(w - D_2)(w - D_3)(w - D_4) = d,$$

which can also be writen in the forms

$$\frac{1}{w - D_1} = \frac{(w - D_2)(w - D_3)(w - D_4)}{d},$$

$$\frac{1}{(w - D_2)(w - D_3)} = \frac{(w - D_1)(w - D_4)}{d},$$

$$\frac{1}{(w - D_1)(w - D_2)(w - D_3)} = \frac{w - D_4}{d}, \text{ etc.}$$

The following notation will be very useful in the sequel.

$$(2.1^*.6) \qquad \begin{cases} f_{i,k}(w) = (w - D_i)(w - D_{i+1}), \ldots, (w - D_k) \\ f_{i,i}(w) = w - D_i. \qquad 1 \le i \le k \le n. \end{cases}$$

Thus, for instance we have from (2.1*.6)

$$f_{1,n-1}(w) = (w - D_1)(w - D_2) \ldots (w - D_{n-1}),$$

$$f_{1,1}(w) = w - D_1.$$

The following formulas will be used constantly. They are obtained from

$$(2.1^*.7) \qquad (w - D_1)(w - D_2) \ldots (w - D_n) - d = 0 \text{ and } (2.1^*.6).$$

$$(2.1^*.8) \qquad \begin{cases} \dfrac{1}{f_{i,k}(w)} = \dfrac{f_{1,i-1}(w) \cdot f_{k+1,n}(w)}{d}; & (i > 1,\ k < n); \\[2ex] \dfrac{1}{f_{1,k}(w)} = \dfrac{f_{k+1,n}(w)}{d}; & (k < n); \\[2ex] \dfrac{1}{f_{i,n}(w)} = \dfrac{f_{1,i-1}(w)}{d}; & (i > 1). \end{cases}$$

The reader should also note that all

$$(2.1^*.9) \qquad w - \mathrm{D}_i = w - \rho^s \mathrm{D} \quad \text{are non zero.}$$

An important step in the development of the (BGEA) of $a^{(0)}$, $a^{(0)}$ from (2.1*.4), is the calculations of the $b_i^{(v)}$ $(i = 1, 2, 3;\ v = 0, 1, \ldots)$. Now by definition of the (BGEA) (in our case) and (2.1.2) (ii) since $[w]` = \mathrm{D}$, where

$$[w]` = \begin{cases} [w]; & for\ d > 0 \\ [w]+1; & for\ d < 0 \end{cases}$$

$$(2.1^*.10) \qquad b_i^{(v)} = a_i^{(v)} \ (\rho^{\mathrm{s}} \mathrm{D}),\ (i = 1, 2, 3;\ v = 0, 1, \ldots)$$

Since $\rho^s\mathrm{D}$ could be any of the numbers D_1, D_2, D_3, D_4, we chose $\rho^{\mathrm{s}}\ \mathrm{D} = \mathrm{D}_1$ so that

$$(2.1^*.11) \qquad b_i^{(v)} = a_i^{(v)}(\mathrm{D}_1),\ i = 1, 2, 3\ ;\ (v = 0, 1, \ldots).$$

Formula (2.1*.11) simply means, that if $a_i^{(v)} = a_i^{(v)}(w)$, which is always the case for algebraic vectors, then $b_i^{(v)}$ is obtained form $a_i^{(v)}(w)$ by substituting D_1 for w.

The following formula, based on (2.1*.11), is important

$$(2.1^*.12) \qquad \begin{cases} If\ a_i^{(v)} = (w - D_1)P,\ P = P(w), \\ (P\ a\ polynomial\ in\ w) \\ then\ b_i^{(v)} = 0 \quad (i = 1, 2, 3;\ v = 0, 1, 2, \ldots). \end{cases}$$

We shall now carry out the (BGEA) for our starting vector (2.1*.4), proceeding in four steps.

Step one Calculating $a^{(v)}$

$$(2.1^*.13) \qquad a^{(v)} = \left(a_1^{(v)},\ a_2^{(v)},\ a_3^{(v)}\right)$$

Step two Calculating $b^{(v)}$

$$(2.1^*.14) \qquad b^{(v)} = \left(a_1^{(v)}(D_1),\ a_2^{(v)}(D_1),\ a_3^{(v)}(D_1)\right).$$

Step three Calculating $a^{(v)} - b^{(v)}$

$$(2.1^*.15) \qquad a^{(v)} - b^{(v)} = \left(a_1^{(v)} - b_1^{(v)},\, a_2^{(v)} - b_2^{(v)},\, a_3^{(v)} - b_3^{(v)}\right).$$

Step four Calculating $a^{(v+1)}$

$$(2.1^*.16) \qquad a^{(v+1)} = \left(a_1^{(v)} - b_1^{(v)}\right)^{-1} \left(a_2^{(v)} - b_2^{(v)},\, a_3^{(v)} - b_3^{(v)},\, 1\right).$$

Taking into account (2.1*.11), (2.1*.12) and (2.1*.5) we obtain

$$a^{(0)} = (f_{1,3}(w),\ f_{1,2}(w),\ f_{2,2}(w)).$$

$$b^{(0)} = (\quad 0, \qquad 0, \ \ \mathrm{D}_1 - \mathrm{D}_2).$$

$$a^{(0)} - b^{(0)} = (f_{1,3}(w),\ f_{1,2}(w),\ f_{1,1}(w)).$$

$$a^{(1)} = \left(\frac{f_{1,2}(w)\cdot f_{4,4}(w)}{d},\ \frac{f_{1,1}(w)\cdot f_{4,4}(w)}{d},\ \frac{f_{4,4}(w)}{d}\right).$$

$$b^{(1)} = \left(\qquad 0, \qquad 0, \qquad \frac{D_1 - D_4}{d}\right).$$

$$a^{(1)} - b^{(1)} = \left(\frac{f_{1,2}(w)\cdot f_{4,4}(w)}{d},\ \frac{f_{1,1}(w)\cdot f_{4,4}(w)}{d},\ \frac{f_{1,1}(w)}{d}\right).$$

$$a^{(2)} = \left(\frac{f_{1,1}(w)\cdot f_{3,4}(w)}{d},\ \frac{f_{1,1}(w)\cdot f_{3,3}(w)}{d},\ f_{3,3}(w)\right).$$

$$b^{(2)} = (\qquad 0\,, \qquad 0\,, \qquad \mathrm{D}_1 - \mathrm{D}_3).$$

and suppressing the arguments w, we have

$$a^{(2)} - b^{(2)} = \left(\frac{f_{1,1}\cdot f_{3,4}}{d},\ \frac{f_{1,1}\cdot f_{3,3}}{d},\ f_{1,1}\right).$$

$$a^{(3)} = \left(\frac{f_{1,3}}{d},\ f_{1,2}\,,\ f_{2,2}\right).$$

Here we note that, if $d = 1$ (but not if $d = -1$) then $a^{(3)} = a^{(0)}$, and we have **Theorem (2.1*.1)**

Let $w = \sqrt[4]{D^4+1}$, $\mathrm{D} \in \mathbb{N}$. Then the (BGEA) of

$$a^{(0)} = \left(f_{1,3}\ ,\ f_{1,2}\ ,\ f_{2,2}\right)$$

with $f_{1,4} = 1$, $b_i^{(v)} = a_i^{(v)}(\mathrm{D}_1)$ $(i = 1, 2, 3;\ v = 0, 1, 2, \ldots)$ is purely periodic and the length of the primitive period is $m = 3$.

We now proceed with the (BGEA) in the case where $d \neq 1$, and obtain further

$$b^{(3)} = (\ 0\ ,\ 0\ ,\ \mathrm{D}_1 - \mathrm{D}_2)$$

$$a^{(3)} - b^{(3)} = \left(\frac{f_{1,3}}{d}\ ,\ f_{1,2}\ ,\ f_{1,1}\right).$$

$$a^{(4)} = (f_{1,2} \cdot f_{4,4}\ ,\ f_{1,1} \cdot f_{4,4}\ ,\ f_{4,4})$$

$$b^{(4)} = (\ 0\ ,\ 0\ ,\ \mathrm{D}_1 - \mathrm{D}_4).$$

$$a^{(4)} - b^{(4)} = (f_{1,2} \cdot f_{4,4}\ ,\ f_{1,1} \cdot f_{4,4}\ ,\ f_{1,1}).$$

$$a^{(5)} = \left(\frac{f_{1,1} \cdot f_{3,4}}{d},\ \frac{f_{1,1} \cdot f_{3,3}}{d},\ \frac{f_{3,3}}{d}\right).$$

$$b^{(5)} = \left(\quad 0, \qquad 0,\ \frac{D_1 - D_3}{d}\right).$$

$$a^{(5)} - b^{(5)} = \left(\frac{f_{1,1} \cdot f_{3,4}}{d},\ \frac{f_{1,1} \cdot f_{3,3}}{d},\ \frac{f_{1,1}}{d}\right).$$

$$a^{(6)} = \left(\frac{f_{1,3}}{d},\ \frac{f_{1,2}}{d},\ f_{2,2}\right).$$

$$b^{(6)} = (\ 0\ ,\ 0\ ,\ D_1 - D_2).$$

$$a^{(6)} - b^{(6)} = \left(\frac{f_{1,3}}{d},\ \frac{f_{1,2}}{d},\ f_{1,1}\right).$$

$$a^{(7)} = \left(\frac{f_{1,2} \cdot f_{4,4}}{d},\ f_{1,1} \cdot f_{4,4}\ ,\ f_{4,4}\right)$$

$$b^{(7)} = (\ 0\ ,\ 0\ ,\ \mathrm{D}_1 - \mathrm{D}_4).$$

$$a^{(7)} - b^{(7)} = \left(\frac{f_{1,2} \cdot f_{4,4}}{d},\ f_{1,1} \cdot f_{4,4}\ ,\ f_{1,1} \right).$$

$$a^{(8)} = (f_{1,1} \cdot f_{3,4}\ ,\ f_{1,1} \cdot f_{3,3}\ ,\ f_{3,3}).$$

$$b^{(8)} = (\quad 0, \quad 0, \quad D_1 - D_3).$$

$$a^{(8)} - b^{(8)} = (f_{1,1} \cdot f_{3,4}\ ,\ f_{1,1} \cdot f_{3,3}\ ,\ f_{1,1}).$$

$$a^{(9)} = \left(\frac{f_{1,3}}{d}, \frac{f_{1,2}}{d}, \frac{f_{2,2}}{d} \right).$$

$$b^{(9)} = \left(0\ ,\ 0\ ,\ \frac{D_1 - D_2}{d} \right)$$

$$a^{(9)} - b^{(9)} = \left(\frac{f_{1,3}}{d}, \frac{f_{1,2}}{d}, \frac{f_{1,1}}{d} \right).$$

$$a^{(10)} = \left(\frac{f_{1,2} \cdot f_{4,4}}{d}, \frac{f_{1,1} \cdot f_{4,4}}{d}, f_{4,4} \right)$$

$$b^{(10)} = (\quad 0, \quad 0, \quad D_1 - D_4).$$

$$a^{(10)} - b^{(10)} = \left(\frac{f_{1,2} \cdot f_{4,4}}{d}, \frac{f_{1,1} \cdot f_{4,4}}{d}, f_{1,1} \right)$$

$$a^{(11)} = \left(\frac{f_{1,1} \cdot f_{3,4}}{d}, f_{1,1} \cdot f_{3,3}\ , f_{3,3} \right)$$

$$b^{(11)} = (\quad 0, \quad 0, \quad D_1 - D_3).$$

$$a^{(11)} - b^{(11)} = \left(\frac{f_{1,1} \cdot f_{3,4}}{d}, f_{1,1} \cdot f_{3,3}\ , f_{1,1} \right)$$

$$a^{(12)} = \left(f_{1,3}\ ,\ f_{1,2}\ ,\ f_{2,2} \right).$$

From the last line we see that

(2.1*.17) $a^{(0)} = a^{(12)}$, and we have thus obtained:

Theorem (2.1*.2)

Let $w = \sqrt[4]{D^4 + d}$, $|d|$, $\mathrm{D} \in \mathbb{N}$, $d \neq 1$, d / D, then (BGEA) of

$a^{(0)} = (f_{1,3}\ ,\ f_{1,2}\ ,\ f_{2,2})$ with $f_{1,4} = d$ and the further notation of Theorem 2.1*.1, is purely periodic and the length of the primitive period is
$m = 12 = 3 \cdot 4$.

Analyzing the vectors $a^{(v)}$, $v = 0, 1, 2, \ldots, 12$ of the (BGEA) of Theorem 2.1*.2 we make the following important observations:

0_1 The twelve vectors of the primitive period of the (BGEA) for the case $n = 4$ consist of four cycles, each containing three vectors, namely a^{3s+t} ; $t = 0,$
1, 2 ;
$s = 0, 1, 2, 3$.

Definition 2.1*.1: A sequence of k - vectors is said to be a cycle of length k if the $(k+1)$-st member is, up to a factor d^{-1}, the same as the starting vector.

0_2 If we disregard the factor d-1 in the components of the vectors we shall call a vector a "modified" vector and denote it $\overline{a}^{(v)}$.

Up to the factor d^{-1}, the components of the vectors in each cycle are the same for a given t, viz.

$$a^{(3s_1+t)} = \overline{a}^{(3s_2+t)}\ ,\quad s_1, s_2 = 0, 1, 2, 3\ ;\ t = 0, 1, 2.$$

0_3 The companion vectors $b^{(v)}$, $(v = 0, 1, \ldots, 11)$ are all integral algebraic vectors, since their components are either 0, $D_1 - D_i$ or $\dfrac{D_1 - D_i}{d}$ $(i = 2, 3, 4)$ and d/D.

Specifically we have $b_i^{(v)} = 0$, $(i = 1, 2)$.

$$b_3^{(v)} = s(D_1 - D_k)\ ,\ s = 1 \text{ or } d^{-1}\ ,\ k = 2, 3, 4.$$

0_4 For three cycles we have

$$a_3^{(3s)} \cdot a_3^{(3s+1)} \cdot a_3^{(3s+2)} = \frac{f_{2,4}}{d}\ ,\quad s = (0, 1, 3)\ ;$$

while for one cycle we have

$$a_3^{(3s)} \cdot a_3^{(3s+1)} \cdot a_3^{(3s+2)} = f_{2,4}\ ,\quad (s = 2)\ ;$$

2.2*. A periodic (BGEA), $n = 5$

We continue the (BGEA) for the case $n = 5$, viz.

(2.2*.1) $$w = \sqrt[5]{D^5 + d}\,,\ D\,,\ |d| \in \mathbb{N}\,,\ d / \mathrm{D} \quad \rho = \exp\left(\frac{2\pi i}{5}\right).$$

As in the previous section it is easy to see that

(2.2*.2) $\mathrm{D} < w < \mathrm{D} + 1\,,\ \ [w]^{`} = \mathrm{D}.$

From (2.2*.1) we have

(2.2*.3) $$\prod_{1}^{5}(w - D_i) = d \ \text{ or } \ f_{1,5} = d.$$

(2.2*.4) $\{\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_4, \mathrm{D}_5\} = \{\mathrm{D}, \rho\,\mathrm{D}, \rho^2\,\mathrm{D}, \rho^3\,\mathrm{D}, \rho^4\,\mathrm{D}\}.$

We shall now carry out the (BGEA) for the starting vector $a^{(0)}$.

(2.2*.5) $a^{(0)} = (f_{1,4}\,,\ f_{1,3}\,,\ f_{1,2}\,,\ f_{2,2}).$

The relations hold.

(2.2*.6) $f_{1,5} = d$ and

(2.2*.7) $w - \mathrm{D}_i = w - \rho^s\,\mathrm{D} \neq 0 \ \ (i = 1, \ldots, 5\,;\ \ s = 0, 1, \ldots, 4).$

By the definition of (BGEA), and with $[w]^{`} = \mathrm{D}$, the $b_i^{(v)}$ are calculated by

(2.2*.8) $b_i^{(v)} = a_i^{(v)}(\rho^s\,\mathrm{D})\,,\ (i = 1, \ldots, 4)\,,\ (v = 0, 1, \ldots).$

and choosing $\rho^s\,\mathrm{D} = \mathrm{D}_1$, we get

(2.2*.9) $b_i^{(v)} = a_i^{(v)}(\mathrm{D}_1)\,,\ (i, v$ as in (2.2*.8)).

The following formula, based on (2.2*.9), is frequently used.

(2.2*.10) Let $a_i^{(v)} = (w - \mathrm{D}_1)\,\mathrm{P}(w)$; (P a polynomial in w)

then $b_i^{(v)} = 0 \ \ (i = 1, \ldots, 4;\ v = 0, 1, \ldots).$

We shall now write down the successive vectors in the (BGEA) of $a^{(0)}$, $a^{(0)}$ from (2.2*.5) and their companion vectors, without carrying out the calculations in the four – step – method, as used in the previous case $n = 4$. These calculations are completely analogus here, and will therefore not be repeated.

$$a^{(0)} = (f_{1,4}\,,\ f_{1,3}\,,\ f_{1,2}\,,\ f_{2,2}).$$

$$b^{(0)} = (0\,,\ 0\,,\ 0\,,\ D_1 - D_2).$$

$$a^{(1)} = \left(\frac{f_{1,3}\cdot f_{5,5}}{d},\ \frac{f_{1,2}\cdot f_{5,5}}{d},\ \frac{f_{1,1}\cdot f_{5,5}}{d},\ \frac{f_{5,5}}{d}\right).$$

$$b^{(1)} = \left(\ 0,\quad 0,\quad 0,\quad \frac{D_1 - D_5}{d}\right).$$

$$a^{(2)} = \left(\frac{f_{1,2}\cdot f_{4,5}}{d},\ \frac{f_{1,1}\cdot f_{4,5}}{d},\ \frac{f_{1,1}\cdot f_{4,4}}{d},\ f_{4,4}\right).$$

$$b^{(2)} = (\ 0\,,\quad 0\,,\quad 0\,,\quad D_1 - D_4).$$

$$a^{(3)} = \left(\frac{f_{1,1}\cdot f_{3,5}}{d},\ \frac{f_{1,1}\cdot f_{3,4}}{d},\ f_{1,1}\cdot f_{3,3}\,,\ f_{3,3}\right).$$

$$b^{(3)} = (\ 0\ ,\quad 0\ ,\quad 0\ ,\quad D_1 - D_3)$$

$$a^{(4)} = \left(\frac{f_{1,4}}{d},\quad f_{1,3}\,,\quad f_{1,2}\ ,\quad f_{2,2}\right)$$

$$b^{(4)} = (\ 0\ ,\quad 0\ ,\quad 0\ ,\ D_1 - D_2).$$

$$a^{(5)} = \left(f_{1,3}\cdot f_{5,5}\,,\ f_{1,2}\cdot f_{5,5}\,,\ f_{1,1}\cdot f_{5,5}\ ,\ f_{5,5}\right).$$

$$b^{(5)} = (\ 0,\quad 0,\quad 0,\ D_1 - D_5).$$

$$a^{(6)} = \left(\frac{f_{1,2}\cdot f_{4,5}}{d},\ \frac{f_{1,1}\cdot f_{4,5}}{d},\ \frac{f_{1,1}\cdot f_{4,4}}{d},\ \frac{f_{4,4}}{d}\right).$$

$$b^{(6)} = \left(\ 0,\quad 0,\quad 0,\ \frac{D_1 - D_4}{d}\right).$$

$$a^{(7)} = \left(\frac{f_{1,1}\cdot f_{3,5}}{d}\quad \frac{f_{1,1}\cdot f_{3,4}}{d}\quad \frac{f_{1,1}\cdot f_{3,3}}{d}\quad f_{3,3}\right)$$

$$b^{(7)} = (\ 0,\quad 0,\quad 0,\ D_1 - D_3).$$

$$a^{(8)} = \left(\frac{f_{1,4}}{d}, \quad \frac{f_{1,3}}{d}, \quad f_{1,2}, \quad f_{2,2} \right).$$

$$b^{(8)} = (\ 0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_2).$$

$$a^{(9)} = \left(\frac{f_{1,3} \cdot f_{5,5}}{d}, \ f_{1,2} \cdot f_{5,5}, \ f_{1,1} \cdot f_{5,5}, \ f_{5,5} \right).$$

$$b^{(9)} = (\ 0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_5).$$

$$a^{(10)} = \left(f_{1,2} \cdot f_{4,5}, \ f_{1,1} \cdot f_{4,5}, \ f_{1,1} \cdot f_{4,4}, \ f_{4,4} \right).$$

$$b^{(10)} = (\ 0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_4).$$

$$a^{(11)} = \left(\frac{f_{1,1} \cdot f_{3,5}}{d}, \ \frac{f_{1,1} \cdot f_{3,4}}{d}, \ \frac{f_{1,1} \cdot f_{3,3}}{d}, \ \frac{f_{3,3}}{d} \right).$$

$$b^{(11)} = \left(\ 0, \quad 0, \quad 0, \quad \frac{D_1 - D_3}{d} \right).$$

$$a^{(12)} = \left(\frac{f_{1,4}}{d}, \quad \frac{f_{1,3}}{d}, \quad \frac{f_{1,2}}{d}, \quad f_{2,2} \right).$$

$$b^{(12)} = (\ 0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_2).$$

$$a^{(13)} = \left(\frac{f_{1,3} \cdot f_{5,5}}{d}, \ \frac{f_{1,2} \cdot f_{5,5}}{d}, \ f_{1,1} \cdot f_{5,5}, \ f_{5,5} \right).$$

$$b^{(13)} = (\ 0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_5).$$

$$a^{(14)} = \left(\frac{f_{1,2} \cdot f_{4,5}}{d}, \ f_{1,1} \cdot f_{4,5}, \ f_{1,1} \cdot f_{4,4}, \ f_{4,4} \right).$$

$$b^{(14)} = (\ 0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_4).$$

$$a^{(15)} = \left(f_{1,1} \cdot f_{3,5}, \ f_{1,1} \cdot f_{3,4}, \ f_{1,1} \cdot f_{3,3}, \ f_{3,3} \right).$$

$$b^{(15)} = (\ 0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_3).$$

$$a^{(16)} = \left(\frac{f_{1,4}}{d}, \ \frac{f_{1,3}}{d}, \ \frac{f_{1,2}}{d}, \ \frac{f_{2,2}}{d} \right).$$

$$b^{(16)} = \left(0, \quad 0, \quad 0, \quad \frac{D_1 - D_2}{d} \right).$$

$$a^{(17)} = \left(\frac{f_{1,3} \cdot f_{5,5}}{d}, \frac{f_{1,2} \cdot f_{5,5}}{d}, \frac{f_{1,1} \cdot f_{5,5}}{d}, f_{5,5} \right).$$

$$b^{(17)} = (0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_5).$$

$$a^{(18)} = \left(\frac{f_{1,2} \cdot f_{4,5}}{d}, \frac{f_{1,1} \cdot f_{4,5}}{d}, \frac{f_{1,1} \cdot f_{4,4}}{d}, f_{4,4} \right)$$

$$b^{(18)} = (0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_4).$$

$$a^{(19)} = \left(\frac{f_{1,1} \cdot f_{3,5}}{d}, f_{1,1} \cdot f_{3,4}, f_{1,1} \cdot f_{3,3}, f_{3,3} \right).$$

$$b^{(19)} = (0, \quad 0, \quad 0, \quad \mathrm{D}_1 - \mathrm{D}_3).$$

$$a^{(20)} = \left(f_{1,4}, \quad f_{1,3}, \quad f_{1,2}, \quad f_{2,2} \right).$$

and from the last line we learn, as could have been expected,

$$(2.2^*.11) \qquad a^{(0)} = a^{(20)}.$$

Taking into account these results, we can now state

Theorem 2.2*.1

Let $w = \sqrt[5]{D^5 + d}$, D, $|d| \in \mathbb{N}$, $d \,/\, \mathrm{D}$, $d \neq 1$. Then the (BGEA) of $a^{(0)}$, where $a^{(0)} = (f_{1,4}, f_{1,3}, f_{1,2}, f_{2,2})$ with $f_{1,5} = d$, $b_i^{(v)} = a_i^{(v)}(\mathrm{D}_1)$, and $\{\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_4, \mathrm{D}_5\}$ a permutation of $\{\mathrm{D}, \rho\,\mathrm{D}, \rho^2\,\mathrm{D}, \rho^3\,\mathrm{D}, \rho^4\,\mathrm{D}\}$, is purely periodic.

The length of the primitive period $m = 20 = 4 \cdot 5$. If $d = 1$, the length of the primitive period is $m = 5$. As in the previous case ($n = 4$) we can make also in the $n = 5$ case the following observations. These will serve as the basis for the main theorem for all $n \geq 2$.

0_1 The twenty vectors of the primitive period of (BGEA) ($n = 5$) consist of five cycles, each containing four vectors, namely

$$a^{(4s+t)}; \ (t = 0, 1, 2, 3\,;\ s = 0, 1, 2, 3, 4).$$

0_2 Up to a factor d^{-1}, the components of the vectors in each cycle are the same for a given t, viz.

$$a^{(4s_1+t)} = \overline{a}^{(4s_2+t)}, \quad s_1, s_2 = 0, 1, 2, 3, 4\,;\ t = 0, 1, 2, 3.$$

0_3 The companion vectors $b^{(v)}$, $(v = 0, 1, \ldots, 19)$ are all integral algebraic vectors, since their components are either 0, or $D_1 - D_i$, or $\frac{D_1 - D_i}{d}$ $(i = 2, 3, 4, 5)$.

Specifically we have $b_i^{(v)} = 0$, $(i = 1, 2, 3)$,

$$b_4^{(v)} = s(D_1 - D_k) \text{ with } s = 1 \text{ or } d^{-1},\ k = 2, 3, 4, 5.$$

0_4 For three cycles we have

$$a_4^{(4s)} \cdot a_4^{(4s+1)} \cdot a_4^{(4s+2)} \cdot a_4^{(4s+3)} = \frac{f_{2,5}}{d}$$

for $s = (0, 1, 2, 4)$

For one cycle only we have

$$a_4^{(4s)} \cdot a_4^{(4s+1)} \cdot a_4^{(4s+2)} \cdot a_4^{(4s+3)} = f_{2,5} \quad \text{for } s = 3.$$

2. 3*. The (BGEA) for the general case $n > 2$.

With the help of Theorems (2.1*.2) and (2.2*.1) we are now able to state the general theorem for $n \geq 2$, $w = \sqrt[n]{D^n + d}$ using the (BGEA) as before. The analogy is now quite obvious. The proof is achieved by induction.

Namely, after having written down $a^{(0)}$ and calculated $a^{(1)}$ we find the structure of the general $a^{(k)}$, $(k = 0, 1, \ldots, n\text{-}2)$ of the first cycle, by induction. Knowing the first cycle, we find, also by induction, the structure at the $(k+1)$-st cycle, hence finally achieve the result. The proof follows exactly the method outlined in the cases $n = 4$ and 5. These considerations lead to

Theorem 2.3*.1

Let

(2.3*.1) $\quad w = \sqrt[n]{D^n + d}$, $n \geq 2$, $D \in \mathbb{N}$, $d \in \mathbb{Z}$, d/D.

Then the (BGEA) of $a^{(0)}$ where

$$(2.3^*.2) \qquad a^{(0)} = (f_{1,\,n-1}(w)\,,\; f_{1,\,n-2}(w)\,,\; \ldots, f_{1,\,2}(w)\,,\; f_{2,2}(w))$$

with

$$(2.3^*.3)$$

$$\begin{cases} f_{1,n}(w) - d = 0,\ \{D_1, D_2, \ldots, D_n\} \textit{ is a permutation} \\ \qquad\qquad\qquad\qquad\qquad\quad \textit{of } \{D, \rho D, \ldots, \rho^{n-1} D\} \\ (s = 0, 1, \ldots, n-1) \textit{ and } b_i^{(v)} = a_i^{(v)}(D_1),\ i = 0, 1, \ldots, n-1 \end{cases}$$

is purely periodic and if d≠1, the length of the primitive period is $m = n\,(n\text{-}1)$. If $d = 1$, the length of the primitive period is $m = n - 1$.

Proof:

Henceforth suppressing the arguments w, we have

$$a^{(0)} = (f_{1,\,n-1}\,,\; f_{1,\,n-2}\,,\; \ldots,\; f_{1,2}\,,\quad f_{2,2}\,)$$

$$b^{(0)} = (\;\; 0 \;\;,\;\; 0 \;\;, \ldots,\;\; 0\;\;, \mathrm{D_1\text{-}D_2}).$$

$$a^{(0)} - b^{(0)} = (f_{1,\,n-1}\,,\; f_{1,\,n-2}\,,\; \ldots,\; f_{1,\,2}\,,\quad f_{1,\,1}\,)$$

$$a^{(1)} = \left(\frac{f_{1,n-2} \cdot f_{n,n}}{d}, \frac{f_{1,n-3} \cdot f_{n,n}}{d}, \ldots, \frac{f_{1,1} \cdot f_{n,n}}{d}, \frac{f_{n,n}}{d} \right)$$

$$b^{(1)} = \left(\qquad 0 \qquad , \qquad 0 \qquad , \ldots, \qquad 0\ ,\ \frac{D_1 - D_n}{d} \right)$$

$$a^{(1)} - b^{(1)} = \left(\frac{f_{1,n-2} \cdot f_{n,n}}{d}, \frac{f_{1,n-3} \cdot f_{n,n}}{d}, \ldots, \frac{f_{1,1} \cdot f_{n,n}}{d}, \frac{f_{1,1}}{d} \right)$$

$$a^{(2)} \left(\begin{array}{c} \dfrac{f_{1,n-3} \cdot f_{n-1,n}}{d}, \dfrac{f_{1,n-4} \cdot f_{n-1,n}}{d}, \ldots, \dfrac{f_{1,1} \cdot f_{n-1,n}}{d}, \\ \dfrac{f_{1,1} \cdot f_{n-1,n-1}}{d}, \qquad f_{n-1,n-1} \end{array} \right)$$

$$b^{(2)} = (\quad 0 \quad , \qquad 0 \qquad , \ldots,\;\; 0 \;\;,\;\; 0 \;\;,\;\; \mathrm{D_1 - D_{n-1}}).$$

$$a^{(2)} - b^{(2)} = \left(\frac{f_{1,n-3} \cdot f_{n-1,n}}{d}, \frac{f_{1,n-4} \cdot f_{n-1,n}}{d}, \ldots, \frac{f_{1,1} \cdot f_{n-1,n}}{d}, \frac{f_{1,1} \cdot f_{n-1,n-1}}{d}, f_{1,1} \right)$$

$$a^{(3)} = \left(\frac{f_{1,n-4} \cdot f_{n-2,n}}{d}, \frac{f_{1,n-5} \cdot f_{n-2,n}}{d}, \ldots, \frac{f_{1,1} \cdot f_{n-2,n}}{d}, \frac{f_{1,1} \cdot f_{n-2,n-1}}{d}, f_{1,1} \cdot f_{n-2,n-2}, f_{n-2,n-2} \right)$$

$$b^{(3)} = (0, 0, \ldots, 0, 0, \mathrm{D}_1 - \mathrm{D}_{n-2})$$

$$a^{(3)} - b^{(3)} = \left(\frac{f_{1,n-4} \cdot f_{n-2,n}}{d}, \frac{f_{1,n-5} \cdot f_{n-2,n}}{d}, \ldots, \frac{f_{1,1} \cdot f_{n-2,n}}{d}, \frac{f_{1,1} \cdot f_{n-2,n-1}}{d}, f_{1,1} \cdot f_{n-2,n-2}, f_{1,1} \right)$$

$$a^{(4)} = \left(\frac{f_{1,n-5} \cdot f_{n-3,n}}{d}, \ldots, \frac{f_{1,1} \cdot f_{n-3,n}}{d}, \frac{f_{1,1} \cdot f_{n-3,n-1}}{d}, f_{1,1} \cdot f_{n-3,n-2}, f_{1,1} \cdot f_{n-3,n-3}, f_{n-3,n-3} \right)$$

We shall prove

Lema 2.3*.1 :

The vector $a^{(k)}$, $k = 2, \ldots, n-2$ has the form

$$a^{(k)} = \left(\frac{f_{1,n-k-1} \cdot f_{n-k+1,n}}{d}, \frac{f_{1,n-k-2} \cdot f_{n-k+1,n}}{d}, \ldots, \frac{f_{1,1} \cdot f_{n-k+1,n}}{d}, \frac{f_{1,1} \cdot f_{n-k+1,n-1}}{d}, f_{1,1} \cdot f_{n-k+1,n-2}, f_{1,1} \cdot f_{n-k+1,n-3}, \ldots, f_{1,1} \cdot f_{n-k+1,n-k+1}, f_{n-k+1,n-k+1} \right) \qquad (2.3^*.4)$$

Proof:

We first count the number of components of $a^{(k)}$ to ensure that this is indeed n -1. The components having the factor d^1 number $(n - k\text{-}1) + 1 = n - k$.

The remaining components which do not have the factor d^1 but have factor $f_{1,1}$ number $n - 2-(n - k +1) +1 = k - 2$. Finally there is the last component, so that the total number of components equals $n\text{-}k + (k\text{-}2) +1 = n\text{-}1$.

We proced to prove Lema 2.3*.1 by induction. Our previous calculations show that Lema 2.3*.1, is correct for $k = 2, 3, 4$ (here we presume that $n \geq 6$). Since every component but the last of the vector $a^{(k)}$ contains the factor $w - \mathrm{D}_1$, while the last (n-1)-th component is $w - \mathrm{D}_{n\text{-}k+1}$, and since by our (BGEA)

$$\mathrm{P}[(w - \mathrm{D}_1), f(w)]\,(w - \mathrm{D}_1) = 0.$$

We obtain from (2.3*.4)

$$(2.3^*.5) \qquad b^{(k)} = (\,0\;,\quad 0\;,\ldots,\;\; 0\;,\;\; \mathrm{D}_1 - \mathrm{D}_{n\text{-}k+1}).$$

$$(2.3^*.6)\;\; a^{(k)} - b^{(k)} = \left(\frac{f_{1,n-k-1}\cdot f_{n-k+1,n}}{d}, \frac{f_{1,n-k-2}\cdot f_{n-k+1,n}}{d}, \ldots, \right.$$
$$\frac{f_{1,1}\cdot f_{n-k+1,n}}{d}, \frac{f_{1,1}\cdot f_{n-k+1,n-1}}{d}\;,\;\; f_{1,1}\cdot f_{n-k+1,n-2},$$
$$\left. f_{1,1}\cdot f_{n-k+1,n-3},\ldots,\;\; f_{1,1}\cdot f_{n-k+1,n-k+1},\;\; f_{1,1}\right)$$

From (2.3*.6) we learn that

$$(2.3^*.7) \qquad a_1^{(k)} - b_1^{(k)} = \frac{f_{1,n-k-1}\cdot f_{n-k+1,n}}{d}$$

Hence by definition of the (BGEA) we obtain

$$a^{(k+1)} = \left(f_{1,n-k-1}\cdot f_{n-k+1,n}\right)^{-1}\left(f_{1,n-k-2}\cdot f_{n-k+1,n},\, f_{1,n-k-3}\cdot f_{n-k+1,n},\ldots,\right.$$
$$f_{1,1}\cdot f_{n-k+1,n},\, f_{1,1}\cdot f_{n-k+1,n-1},\; d\, f_{1,1}\cdot f_{n-k+1,n-2},$$
$$\left. d\, f_{1,1}\cdot f_{n-k+1,n-3},\ldots,\; d\, f_{1,1}\cdot f_{n-k+1,n-k+1},\; d\, f_{1,1},\, d\right)$$

$$(2.3^*.8) \qquad a^{(k+1)} = \left(\frac{1}{f_{n-k-1,\,n-k-1}}, \frac{1}{f_{n-k-2,\,n-k-1}}, \ldots, \right.$$
$$\frac{1}{f_{2,n-k-1}}, \frac{1}{f_{2,n-k-1}\cdot f_{n,n}}, \frac{d}{f_{2,n-k-1}\cdot f_{n-1,n}},$$

$$\frac{d}{f_{2,n-k-1}\cdot f_{n-2,n}}, \dots, \frac{d}{f_{2,n-k-1}\cdot f_{n-k+2,n}},$$
$$\frac{d}{f_{2,n-k-1}\cdot f_{n-k+1,n}}, \frac{d}{f_{1,n-k-1}\cdot f_{n-k+1,n}}\Bigg).$$

Rationalizing the denominators of the fractional components in (2.3*.8) we obtain

$$(2.3^*.9)\qquad a^{(k+1)} = \Bigg(\frac{f_{1,n-k-2}\cdot f_{n-k,n}}{d}, \frac{f_{1,n-k-3}\cdot f_{n-k,n}}{d}, \dots,$$
$$\frac{f_{1,1}\cdot f_{n-k,n}}{d}, \frac{f_{1,1}\cdot f_{n-k,n-1}}{d}, f_{1,1}\cdot f_{n-k,n-2},$$
$$f_{1,1}\cdot f_{n-k,n-3}, \dots, f_{1,1}\cdot f_{n-k,n-k}, f_{n-k,n-k}\Big) =$$
$$\Bigg(\frac{f_{1,n-(k+1)-1}\cdot f_{n-(k+1)+1,n}}{d}, \frac{f_{1,n-(k+1)-2}\cdot f_{n-(k+1)+1,n}}{d},$$
$$,\dots, \frac{f_{1,1}\cdot f_{n-(k+1)+1,n}}{d}, \frac{f_{1,1}\cdot f_{n-(k+1)+1,n-1}}{d},$$
$$f_{1,1}\cdot f_{n-(k+1)+1,\,n-2}, f_{1,1}\cdot f_{n-(k+1)+1,\,n-3}, \dots,$$
$$f_{1,1}\cdot f_{n-(k+1)+1,\,n-(k+1)+1}, f_{n-(k+1)+1,\,n-(k+1)+1}\Big).$$

Comparing formulas (2.3*.4) and (2.3*.9) we see that the latter is simply the former with $k+1$ in place of k. Of course, the number of components having the factor d^1 is decreased by one. The structure of $a^{(k)}$ also shows that n -2 is an upper bound for k. Namely, the first component contains the factor $f_{1,\,n-k-1}$. Hence n - k -1, i.e. $n - 2 \geq k$. With $k = n - 2$ in(2.3*.4) we obtain

$$(2.3^*.10)\qquad a^{(n-2)} = \Bigg(\frac{f_{1,1}\cdot f_{3,n}}{d}, \frac{f_{1,1}\cdot f_{3,n-1}}{d}, f_{1,1}\cdot f_{3,n-2}, f_{1,1}\cdot f_{3,n-3},$$
$$,\dots, f_{1,1}\cdot f_{3,3}, f_{3,3}\Big).$$

From (2.3*.10) we obtain as before

$$b^{(n-2)} = (\ 0,\ 0,\ \dots,\ 0,\ \mathrm{D}_1 - \mathrm{D}_3),$$

and from (2.3*.10) and $b^{(n-2)}$ we obtain

$$(2.3^*.11) \qquad a^{(n-2)} - b^{(n-2)} = \left(\frac{f_{1,1}\ f_{3,n}}{d}, \frac{f_{1,1}\ f_{3,n-1}}{d}, f_{1,1}\ f_{3,n-2}, f_{1,1}\ f_{3,n-3}, \ldots, f_{1,1}\ f_{3,3}, f_{1,1} \right).$$

Also (2.3*.11) gives, by definition of the (BGEA),

$$a^{(n-1)} = f_{3,n}^{-1}(f_{3,n-1},\ d f_{3,n-2},\ d f_{3,n-3},\ \ldots,\ d f_{3,3},\ d,\ d/f_{1,1})$$

So

$$(2.3^*.12) \qquad a^{(n-1)} = \left(\frac{f_{1,n-1}}{d}, f_{1,n-2}, f_{1,n-3}, \ldots, f_{1,3}, f_{1,2}, f_{2,2} \right)$$

Equation (2.3*.12) is a most important formula, and as we shall soon see, it almost conatins implicitly the proof of the Main Theorem. Indeed, if we disregard the factor d^{-1} in the components of the vectors $a^{(1)}, a^{(2)}, \ldots, a^{(n-1)}$, the vector so obtained will be called the "modified vector" denoted $\overline{a}^{(\nu)}$.

Thus on comparing (2.3*.2) with (2.3*.12) we see that

$$(2.3^*.13) \qquad a^{(0)} = \overline{a}^{(n-1)}$$

If $d = 1$, then $a^{(0)} = a^{(n-1)}$ and (BGEA) is purely periodic with primitive period length $m = n - 1$.

Let us now presume $d \neq 1$. A simple argument tells us that the construction of the components of the vectors in the sequence $\{a^{(\nu)}\}$ is independent of d, and this means by virtue of (2.3*.13)

$$(2.3^*.14) \qquad \begin{cases} a^{(1)} = \overline{a}^{((n-1)+1)} \\ a^{(2)} = \overline{a}^{((n-1)+2)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^{(n-2)} = \overline{a}^{((n-1)+n-2)} \\ a^{(0)} = \overline{a}^{((n-1)+n-1)} = \overline{a}^{(2(n-1))} \\ a^{(1)} = \overline{a}^{(2(n-1)+1)} \\ a^{(2)} = \overline{a}^{(2(n-1)+2)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^{(n-2)} = \overline{a}^{(2(n-1)+n-2)} \\ a^{(0)} = \overline{a}^{(2(n-1)+n-1)} = \overline{a}^{(3(n-1))} \\ a^{(1)} = \overline{a}^{((3n-1)+1)} \end{cases}$$

Thus, from formula (2.3*.14) we learn that the sequence of the (BGEA) of $a^{(0)}$ proceeds in cycles of (n -1) vectors which, if regarded as modified vectors, are corespondingly equal, viz.

$$(2.3^*.15) \qquad a^{(i)} = \overline{a}^{(j(n-1)+i)} \qquad (i = 0, 1, \ldots, n-2;\ j = 1, 2, \ldots).$$

Thus in order to get the complete structure of the vectors $a^{(v)}$ of the (BGEA) of $a^{(0)}$, we still have to investigate the distribution of the factors d^{-1} in the components of the vectors $a^{(v)}$. Since we have only n -1 different modified vectors, the total number of different vectors in the sequence $\{a^{(v)}\}$ of (BGEA) of $a^{(0)}$ is finite, and the (BGEA) of $a^{(0)}$ is periodic. In order to find the length of the primitive period, we first prove:

Lemma 2.3*.2.

If in a vector $a^{(v)}$ the first k components ($1 \le k \le n$ -1) have the factor d^{-1} and the remaining (n -1- k) components do not, then in the succeeding vector $a^{(v+1)}$ the first (k-1) components have the factor d^{-1}, and the remaining n - k do not.

The proof of Lemma 2.3*.2. is obvious and can be read off from the proof of Lemma (2.3*.1) taking into account formula (2.1*.8).

We shall now make use of Lemma 2.3*.2 to determine finally the distribution of the factor d^{-1} in the components of $a^{(v)}$.

In $a^{(0)}$ no components have the factor d^{-1}.

In $a^{(n-1)}$ the first component only has the factor d^{-1}.

In $a^{2(n-1)}$ the first two components only have the factor d^{-1}.

In $a^{3(n-1)}$ the first three components only have the factor d^{-1}.

⋮

In $a^{(n-1)(n-1)}$ the first (n-1) components only have the factor d^{-1}.

In $a^{n(n-1)}$ no component has the factor d^{-1}.

Thus we have obtained

$$(2.3^*.16) \qquad m = \begin{cases} n(n-1) & if \quad d \neq 1 \\ n-1 & if \quad d = 1. \end{cases}$$

Now, as we have pointed out, in the vector

$a^{((n-2)(n-1))}$ the first (n - 2) components have the factor d^{-1},

in $a^{((n-2)(n-1)+1)}$ the first (n -3) components have the factor d^{-1},

in $a^{((n-2)(n-1)+2)}$ the first (n -4) components have the factor d^{-1},

$\vdots$

in $a^{((n-2)(n-1)+n-3)}$ the first component only has the factor d^{-1},

in $a^{((n-2)(n-1)+n-2)}$ no component has the factor d^{-1}.

Thus, in the sequence of (n -1) vectors

$$(2.3^*.17) \qquad \{a^{(v)}\},\ v = (n-2)(n-1) + r,\ 0 \leq r \leq n-2,$$

the last component never has the factor d^1. A simple argument shows that in every other sequence of n-1 vectors, only one vector has in the last component the factor d^1. The product of the last components in each of the successive components of a class of (n -1) successive vectors can be easly verified to be

$$(2.3^*.18) \qquad \frac{(w-D_2)(w-D_3)....(w-D_n)}{d} \text{ or } \frac{f_{2,n}}{d}$$

with the exception of the calss (2.3*.17). This product is then

$$(w-D_2)(w-D_3)....(w-D_n)$$

so that the product of all last components of all the vectors of the primitive period $a^{(v)}$, $v = 0, 1, \ldots, n(n-1)-1$ equals

$$(2.3^*.19) \qquad \frac{[(w-D_2)(w-D_3)....(w-D_n)]^n}{d^{n-1}}.$$

Thus we have proved the Main Theorem and verified all observations stated there.

Concluding we want to point out that all companion vectors at the vectors of the primitive period have the form

$$b^{(v)} = (0,\ 0,\ \ldots,\ 0,\ \mathrm{D}_1 - \mathrm{D}_i),\ \ (i = 2, \ldots, n)$$

or

$$b^{(t)} = \left(0,\ 0,\ \ldots,\ 0,\ \frac{D_1 - D_j}{d}\right), \quad (j = 2, \ldots, n)$$

and since $d \,/\, \mathrm{D}_1 - \mathrm{D}_j$ all the components of all the vectors of the primitive period are algebraic integers.

With Theorem (2.3*.1) we have achieved our main aim, namely, we have constructed an algorithm by which a vector $a^{(0)} \in \mathrm{Q}(w)^{n-1}$ of dimension $(n-1)$, $w = \sqrt[n]{D^n + d}$, $d \in \mathbb{Z}$, $\mathrm{D} \in \mathbb{N}$, d/D becomes periodic without the restriction $\mathrm{D} \geq (n-2)d$ for $d > 0$ or $\mathrm{D} \geq 2(n-1)|d|$ for negative d. It is important to note for further purposes that the $b_i^{(v)}$ of the algorithm we have constructed are all algebraic integers.

We concluded this section by showing that in the case $n = 2$, (BGEA) coincides with continued fractions.

$$\text{(2.3*.20)} \qquad \begin{cases} w = \sqrt{D^2 + d} \ \ d \,/\, D,\ d \in \mathrm{Z} \ \ and \ \ D \in N. \\ a^{(0)} = w + D \end{cases}$$

In this case (as in any other case of a real quadratic irrational) the (JPA) becomes the Euclidean Algorithm which gives the expansion of $a^{(0)}$ as a simple continued (periodic) fraction.

By the definition of (JPA) we have here

$$\text{(2.3*.21)} \qquad \begin{cases} a^{(v+1)} = \dfrac{1}{a^{(v)} - b^{(v)}} \ (1) = \dfrac{1}{a^{(v)} - b^{(v)}} \\ a^{(v)} = b^{(v)} + \dfrac{1}{a^{(v+1)}} \ ;\ b^{(v)} = \left[a^{(v)}\right] \ v = 0, 1, \ldots \\ a^{(0)} = b^{(0)} + \cfrac{1}{b^{(1)} + \cfrac{1}{b^{(2)} + \ldots}} \end{cases}$$

$$a^{(0)} = (w + \mathrm{D})$$

$$[w] = \mathrm{D} \quad b^{(0)} = 2\mathrm{D}$$

$$a^{(1)} = \frac{1}{a^{(0)} - b^{(0)}} = \frac{1}{w - D} = \frac{w + D}{d}$$

$$b^{(1)} = \frac{2D}{d}$$

$$a^{(2)} = \frac{1}{a^{(1)} - b^{(1)}} = \frac{d}{w - D} = w + D$$

Using the (BGEA) in the case n = 2 for (2.3*.5) we have, using

$$w^2 - \mathrm{D}^2 = d\,,$$

$$(w - \mathrm{D})\,(w + \mathrm{D}) = d\,; \quad -\mathrm{D} = -\mathrm{D}_1\,; \quad +\mathrm{D} = -\mathrm{D}_2.$$

$$(w - \mathrm{D}_1)\,(w - \mathrm{D}_2) = d$$

$$b^{(v)} = a^{(v)}\,(\mathrm{D}_1).$$

$$a^{(0)} = w + \mathrm{D} = w - \mathrm{D}_2$$

$$b^{(0)} = \mathrm{D}_1 - \mathrm{D}_2$$

$$a^{(0)} - b^{(0)} = \mathrm{w} - \mathrm{D}_1.$$

$$a^{(1)} = \frac{1}{w - D_1} = \frac{w - D_2}{d}\,.$$

$$b^{(1)} = \frac{D_1 - D_2}{d}\,.$$

$$a^{(1)} - b^{(1)} = \frac{w - D_1}{d}\,.$$

$$a^{(2)} = \frac{1}{a^{(1)} - b^{(1)}} = \frac{d}{w - D_1} = w - D_2 = a^{(0)}\,.$$

as it should be by Theorem 2.3*.1.

As is well known (HBA) applied to a vector $a^{(0)}$, $a^{(0)} \in \mathrm{Q}(w)^{n-1}$,

$$w = \sqrt[n]{D^n + d}\,, \quad d \in \mathbb{Z},\ \mathrm{D} \in \mathbb{N},\ \ d\,/\,\mathrm{D},$$

becomes periodic with the restriction

$$\mathrm{D} \geq (n-2)d \ \text{ for positive } d$$

or

$$\mathrm{D} \geq 2(n-1)|d| \ \text{ for negative } d.$$

The advantage of (BGEA) is that this algorithm applied to a vector

$a^{(0)} \in Q(w)^{n-1}$, $w = \sqrt[n]{D^n + d}$, $d \in \mathbb{Z}$, $D \in \mathbb{N}$, d/D, becomes periodic without the restriction on D.

Sections 2.2. The proof of the sufficient condition for the restricted periodicity of (BGEA).

In that section we will prove that *d*/D cannot be eliminated in proving the priodicity of (BGEA), and this will prove that *d*/D is also the sufficient condition in proving (BGEA) restrictive periodic, thus characterizing (BGEA) as the Generalized Euclidean Algorithm.

Hilbert realized that (EA) is a very powerful algorithm because it is unrestrictively periodic and as a results of its unrestricted periodicity many important problems in quadratics or E^2 quoted in Section 1.2. were completely solved.

The same famous perob1ems in *n*-dimensions or E^n remained open questions in the Algebraic Number Theory.

These problems are:

(2.2.1) To prove the one to one correspondence between the real numbers and the oriented straight line.

(2.2.2) Hermit's problem to find a periodic algorithmic development for higher degree irrationals.

(2.2.3) Solutions for higher degree Diophantine Equations.

(2.2.4) The problem to find the multiplicative group of units in higher degree algebraic number fields (Dirichlet's problem for any *n*).

(2.2.5) The existence of an algorithm to approximate higher degree irrationals once the Hilbert Completeness Axiom was accepted.

(2.2.6) To find relations between roots and coefficients for higher degree polynomials, as related to Galois' Theory of polynomials.

(2.2.7) Ability to prove Fermat's last theorem to show that no positive integer solutions exist for $x^n + y^n = z^n$ for $n > 2$.

All of those open questions for $n > 2$ caused Hilbert to ask for the invention of a universal algorithm as powerful as (EA) for $n = 2$ in order to solve all of the previously mentioned problems in higher dimensions from the periodicity of this universal algorithm.

This Hilbert "Zahlbericht" is known as Hilbert's 10-th Problem.

Logicians proved that Hilbert's hoped - for algorithm which would be unrestrictedly periodic and which would solve all of these above mentioned problems from ist periodicity, does not exists.

No explicit solution of Hilbert's 10-th Problem was given.

In his 10-th Problem Hilbert asked for the invention of the Generalized Euclidean Algorithm (GEA) and for the n-dimensional equivalent of (n-ELT) form quadratics to prove its unrestricted periodicity.

We use this result, proved by the logicians (who did not provide an explicite proof), concerning Hilbert's 10-th Problem to prove that d/D is also a sufficient condition in proving (BGEA) restricted periodicity.

By logic it was proved that Hilbert's hoped – for periodic algorithm does not exist if it is required to be always periodic.

If d/D in the periodicity of (BGEA) could be eliminated then it would contradict the resolution of Hilbert's 10-th Problem by the logicians, and therefore the restriction d/D cannot be eliminated in proving (BGEA) periodic and as such (BGEA) now, is proved to be restrictiv periodic. It is true that if d/D is not valid, then (BGEA) is not periodic since otherwise it will contradict Hilbert's 10-th Problem.

In [13] we completed the proof of the restricted periodicity of (BGEA). Since (BGEA) is of the same cut or prototype as (EA), then (BGEA) is the only Generalized Euclidean Algorithm (GEA).

Section 2. 3. (BGEA) and *n*-dimensional equivalent of Euler - Lagrange Theorem (*n*-ELT).

In 1907 Perron [40] was successful in showing that if a development is periodic then the components of the initial vector are algebraic numbers.

In 1980 and 1995 Baica [1,13] proved the restricted periodicity of

(BGEA). With these two important results we can state Perron- Baica's Theorem (PBT) as follows:

Perron – Baica Theorem 2.3.1.

Every n-degree irrationals w which can be written as $w = \sqrt[n]{D^n + d}$ with $D \in \mathbb{N}$, $d \in \mathbb{Z}$, $n \geq 3$ where d / D makes (BGEA) periodic. Therefore we have a one to one correspondence between irrationals $w > 1$ and infinite (BGEA) sequences.

The sequence is (BGEA) restrictive periodic if and only if w is algebraic of degree n (root, of an irreductible rational polynomial of degree n) of the form

$$w = \sqrt[n]{D^n + d}\ ,\ D \in \mathbb{N},\ d \in \mathbb{Z},\ n \geq 3 \text{ and } d / D.$$

The proof in Baica's direction was presented in Chapter 2, Section 2.1. and 2.2.

The proof in Perron's direction was given by Perron [40] in 1907.

Perron - Baica Theorem (PBT) is therefore the n-dimensional equivalent (n-ELT) of Euler - Lagrange Theorem (2-ELT) from quadratics.

Section 2.4. (BGEA) is the Euler System for the Algebraic Number Theory in *n*-dimensions.

In a previous book MB - 15/PAMM the author [20, 23] proved that (BGEA) is the Euler System for the Algebraic Number Theory in n-dimensions by showing that the (BGEA) solves by means of its restricted periodicity all the following open n-dimensional problems:

(2.4.1) General simple continued fractions algorithm known as Hermite's Problem.

(2.4.2) An n-dimensional equivalent of Euler - Lagrange Theorem from quadratics (n - ELT).

(2.4.3) Dirichlet's Problem.

(2.4.4) The solution of Galois' Theory of polynomials problem.

(2.4.5) An algorithmic approximation of irrationals.

(2.4.6) n-dimensional Fibonacci Numbers.

(2.4.7) The original Euclidean Fermat's Last Theorem Problem (EFLT).

(2.4.8) The only algorithmic explicit solution of Hilbert's 10 Problem.

In [23] the author showed that because of this (BGEA) is the (ES) in the Algebraic Number Theory of $(E^n G)$.