Chapter III. THE ORIGINAL FERMAT'S LAST THEOREM STATED IN EUCLIDEAN TERMS (EFLT) AND THE ELLIPTICAL FERMAT'S LAST THEOREM (ELFLT)

Section 3.0. Introduction

In [15] the author expressed her genuine concern about the proof of Fermat's Last Theorem in the geometry of the elliptic curve (ELFLT) which may not be equvalent to (let alone the same as) the result in the Euclidean Geometry where Fermat's Last Theorem originated (EFLT) about three hundred fifty years ago.

In 1940, E. Schmidt was the first to say that in mathematics, a problem can be approached geometrically, algebraically or analytically and proofs can be given in different mathematical varieties, which are defined in modern mathematics as Mathematical Models. Also, he said that since we cannot compute in a geometry, we have to construct an algebra for that corresponding geometry. He is the father of the General Algebraic Geometry as C.F. Gauss is the father of the Algebraic Number Theory, which is the algebra of the n-dimensional Euclidean Geometry (E^n G).

E. Schmidt was the doctoral father of Jürgen Schmidt who, together with Helmut Hasse, was the doctoral father of the author at the University of Houston, where she earned her Ph.D. in Mathematics (Algebraic Number Theory and Universal Algebra) in 1980.

Section 3.1. The Euclidean character of the original Fermat's Last Theorem

In mathematics we can construct as many geometries or geometrical models as we please. All that we need is to have the elements declared, to state the axioms and the definitions, and to have consistency in our mathematical logic. All of these many geometries do not relate to each other, but they all relate to the topology. Because of this, if you prove something in one geometry it may NOT be the same as in another geometry. Only one geometry is the Euclidean Geometry (EG), the other geometries are non-euclidean geometries. For example if we consider the V-*h* postulate in (EG) where two parallel line do not intersect, they intersect at two ideal points Ω and Ω' in the hyperbolic geometry (HYG).

For the (EG) the elements are points and the straight lines, for the geometry of eliptic curves (GEC) the elements are points and the elliptic curves.

We have to make distinction between what is an element in a geometry and what is a definition in a geometry.

We know that no „Geometry of the elliptic curves" (GEC) and no „Arithmetic of the elliptic curves" known as Hecke and Langlands algebra (HLA), which is the number theory of the (GEC), existed 350 years ago, and we are forced to recognize the strong Euclidean Character of the original Fermat's Last Theorem (EFLT).

Therefore we have to follow Hasse's advice to solve (EFLT) in the Algebraic Number Theory which is the algebra of the *n*-dimensional Euclidean Geometry (E^n G).

Section 3.2. Explicit solutions for Hasse's Diophantine equations $a^2 \pm a\,b + b^2 = c^2$.

In this section we will solve explicity the quadratic diophantine equations of the form.

(3.2.1) $$a^2 \pm a\,b + b^2 = c^2$$

Hasse [33] gave the totality of solutions of (3.2.1) in parameter form and they are known as Hasse equations.

No explicit solutions of $a^2 + a\,b + b^2 = c^2$, $a + b > c > b > a$; $a, b, c \in \mathbb{N}\setminus\{0\}$ and $a^2 - a\,b + b^2 = c^2$, $b > c > a > 0$, $a, b, c \in \mathbb{N}\setminus\{0\}$, were known until the author [7] observed that these are homogeneos quadratic diophantine equations and with a proper linear transformation, they can be reduced to a simple diophantine equation which can be solved explicitly.

In what follows we relate the explicit solutions of (3.2.1) to triples of rational pythagorean triangles (abbr. RPT) having equal areas.

A new method of solving $a^2 + a\,b + b^2 = c^2$ is to set $a = y - 1$, $b = y + 1$, $y \in \mathbb{N} \setminus \{0, 1\}$ and get Euler – Pell's equation $c^2 - 3\,y^2 = 1$.

To solve $a^2 - a\,b + b^2 = c^2$, we set $a = \frac{1}{2}(y + 1)$, $b = y - 1$, $y \geq 2$, $y \in \mathbb{N}$

and get a corresponding Euler-Pell's equation. The infinite number of solutions in Euler – Pell's equation geves rise to an infinity of solutions to $a^2 \pm a\,b + b^2 = c^2$. From this fact the following theorems are proved.

Theorem 3.2.1.

Let $c^2 = a^2 + a\,b + b^2$, $a + b > c > b > a > 0$, then the three RPT-s formed by (c, a), (c, b), $(a + b, c)$ have the same area $S = abc\,(b + a)$ and there are infinitely many such triples of RPT.

Theorem 3.2.2.

Let $c^2 = a^2 - a\,b + b^2$, $b > c > b > a > 0$, then the three RPT-s formed by (b, c), (c, a), $(c, b - a)$ have the same area $S = abc\,(b - a)$ and there are infinitely many such triples of RPT.

3.1*. Definitions and previous results.

In one of his papers Bernstein [28] returned to the greek classical mathematics, and investigated primitive rational Pythagorean Triangles concerning mainly *k*-tuples of them having equal perimeters 2P.

In this paper we deal with rational integral right triangles having equal areas and give the following

Definition 3.1*.1.

A rational triangle with sides *a*, *b*, *c* which are represented by a triple (a, b, c) of natural numbers will be called a Rational Pythagorean Triangle, if and only if there exists

$$(3.1^*.1)\qquad \left.\begin{array}{l}(u,v),(u,v)\in N^2-\{(0,0)\},\\ u>v,\ such\ that\\ a=u^2-v^2,\ b=2uv,\ c=u^2+v^2\\ a,b,c\in N-\{0\}=1,2,\ldots\end{array}\right\}$$

We abreviate Rational Pythagorean Triangle by RPT and we write RPT (u, v) for RPT formed by (u, v). We also write $S(u, v)$ for the area and $P(u, v)$ for half perimeter of the RPT (u, v)

$$(3.1^*.2)\qquad D=RPT(u,v):S(u,v)=\frac{1}{2}\,a\,b=u\,v(u^2-v^2)$$

$$P(u,v)=\frac{1}{2}\,(a+b+c)=u\,(u+v)$$

The main question answered here is to find triples of RPT-s haning equal areas.

The firs who asked this question was the great Diophantus [30] and Dikson [29] enlarged the topic.

Let D be a triangle with integral sides and $\hat{C}=120°$ one of its angles. Then if c is the side opposite $\hat{C}$ and a, b the two adjacent sides of $\hat{C}$, we have, by $c^2=a^2+b^2-2ab\cos\hat{C}$.

$$(3.1^*.3)\qquad \begin{cases}a^2+ab+b^2=c^2\\ a+b>c>b>a;\ a,b,c\in N-\{0\}.\end{cases}$$

and if $\hat{C}=60°$ we have

$$(3.1^*.4)\qquad \begin{cases}a^2-ab+b^2=c^2\\ b>c>a>0;\ a,b,c\in N-\{0\}.\end{cases}$$

The totality of solutions to $a^2\pm a\,b+b^2=c^2$ is given by parameter form by Hasse [33]. The new idea in this paper rests in the fact that (3.1*.3) and (3.1*.4) are connected with the areas of the triangles. In order to find a formula to derive explicity the infinity of RPT-s of equal area, since we cannot use Hasse's [33] parametric form, we will give a new method to prove that the equations $a^2\pm a\,b+b^2=c^2$ have infinitely many solutions and state some of them explicitly.

The new method will bring use to the solution of a Euler – Pell's equation. The infinite number of solutions of Euler – Pell's equation [7] will give rise to an infinity of solution to $a^2 \pm a\, b + b^2 = c^2$.

3.2*. Euler – Pell's equation.

$u_n^2 - 3\,v_n^2 = 1$; $n = 0,\ 1, \ldots$ In the sequel we will permanently have to make use of Euler – Peel's equation

$$u_n^2 - 3\,v_n^2 = 1;\ n = 0, 1, \ldots \qquad (3.2^*.1)$$

This could be solved by continued fraction with $\sqrt{3}+1 = [\,\overline{2,1}\,]$, but we use a simpler method since a solution of (3.2*.1) is easly found.

Neglecting $(u_0,\ v_0) = (1,\ 0)$ we have

$$(u_1,\ v_1) = (2,\ 1), \qquad (3.2^*.2)$$

Hence using [7]

$$u_n + v_n\sqrt{3} = (2 + \sqrt{3}\,)^n,\ n = 0, 1, \ldots . \qquad (3.2^*.3)$$

From (3.2*.3) we deduce

$$\begin{cases} u_n = \sum\limits_{i=0}^{\left[\frac{n}{2}\right]} \binom{n}{2i} 2^{n-2i}\, 3^i\ ;\ n = 0, 1, \ldots \\[2ex] v_n = \sum\limits_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} 2^{n-1-2i}\, 3^i\ ;\ n = 1, 2, \ldots \end{cases} \qquad (3.2^*.4)$$

From (3.2*.4) we obtain

$$\begin{cases} u_{2m} = \sum\limits_{i=0}^{m} \binom{2m}{2i} 2^{2m-2i}\, 3^i\ ;\ m = 1, 2, \ldots \\[2ex] v_{2m} = \sum\limits_{i=0}^{m-1} \binom{2m}{2i+1} 2^{2m-1-2i}\, 3^i\ ;\ m = 1, 2, \ldots \\ (u_0,\ v_0) = (1,\ 0) \end{cases} \qquad (3.2^*.5)$$

$$(3.2^*.6)\qquad \begin{cases} u_{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{2i} 2^{2m+1-2i} 3^i \,;\ m = 0, 1, \ldots \\ \\ v_{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{2i+1} 2^{2m-2i} 3^i \,;\ m = 0, 1, \ldots \end{cases}$$

In u_{2m} all summands are even but the last which is 3^m; in v_{2m} all summands are even, hence

$$(3.2^*.7)\qquad \begin{cases} u_{2m} = 2F + 1 \\ v_{2m} = 2G; \quad F, G \in N \end{cases}$$

In u_{2m+1} all summands are even; in v_{2m+1} all summands are even but the last which is 3^m; hence

$$(3.2^*.7a)\qquad \begin{cases} u_{2m+1} = 2S; \\ v_{2m+1} = 2T + 1; \quad S, T \in N. \end{cases}$$

We have the initial values

$$(3.2^*.8\ \begin{cases} (u_0, v_0) = (1,0); (u_1, v_1) = (2,1); (u_2, v_2) = (7,4) \\ (u_3, v_3) = (26,15); (u_4, v_4) = (97,56); (u_5, v_5) = (362,209) \\ (u_6, v_6) = (1351,780); (u_7, v_7) = (5092,2911). \end{cases}$$

3.3*. A solution of (3.1*.3) $a^2 + ab + b^2 = c^2$.

Here we shall give infinitely many solutions of (3.1*.3) in explicit form, though they won't constitute all solutions of (3.1*.3). Let

$$(3.3^*.1)\qquad \begin{cases} a^2 + ab + b^2 = c^2;\ a + b > c > b > a > 0 \\ a, b, c \in N - \{0\}. \end{cases}$$

We set

$$(3.3^*.2)\qquad a = y - 1;\ b = y + 1;\ y \in \mathbb{N} - \{0\}.$$

Substituting the values of a, b from (3.3*.2) in (3.3*.1) we obtain

$$(y-1)^2 + (y^2 - 1) + (y+1)^2 = c^2$$

or

$$c^2 - 3y^2 = 1.$$

Now we set $c = u_n,\ y = v_n,\ n = 1, 2, \ldots$ and we get $u_n^2 - 3v_n^2 = 1$ or

$$v_n^2 - 3u_n^2 = 1 \text{ which is (3.2*.1).}$$

The infinite number of solutions u_n, v_n in Euler – Pell's equation (3.2*.1), give rise to a infinity of solutions to $a^2 + a\,b + b^2 = c^2$.

Theorem 3.3*.1.

Let $c^2 = a^2 + a\,b + b^2$, $a + b > c > b > a > 0$, then the three RPT-s formed by (c, a), (c, b), $(a + b, c)$ have the same area $S = abc\,(b + a)$ and there are infinitely many such triples of RPT-s.

Proof:

To get the example given by Diophantus [30, p.172] as a particular case of our formulas we consider $(a, b, c) = 1$. Since we prefer $(a, b, c) = 1$, though this is not absolutely necessary, we have to set v_n even in (3.2*.1), so that a, b are both odd. If $v_n = v_{2m}$ we obtain

(3.3*.3) $\qquad a = v_{2m} - 1;\ b = v_{2m} + 1;\ c = u_{2m};\ m = 1, 2, \ldots\ldots.$

Using (3.1*.2) and $a^2 + a\,b + b^2 = c^2$ it is easy to show that

$$S(c, a) = S(c, b) = S\,(a + b, c) = abc(b + a).$$

$$D_1 = \text{RPT}\,(c, a) = \text{RPT}\,(u_{2m}, v_{2m} - 1);$$

$$D_2 = \text{RPT}\,(c, b) = \text{RPT}\,(u_{2m}, v_{2m} + 1);$$

$$D_3 = \text{RPT}\,(a + b, c) = \text{RPT}\,(2\,v_{2m}, u_{2m}),$$

and following $S = abc\,(b + a)$ the common area is

$$\text{(3.3*.4)} \qquad S = 2\,u_{2m}\,v_{2m}\left(v_{2m}^2 - 1\right).$$

That there are infinitely many such triples of RPT-s follow from the infinity of solutions of (3.2*.1).

Choosing $m = 1$, we have from (3.2*.8)

$$\text{(3.3*.5)} \qquad \begin{cases} (u_2,\ v_2) = (7, 4);\ u_2 = 7,\ v_2 = 4 \\ c = 7;\ a = 3;\ b = 5 \\ S = 3 \cdot 5 \cdot 7 \cdot 8 = 840 \end{cases}$$

and this is exactly the example given by Diophantus.

If we choose $m = 2$, we have

$$(3.3^*.6)\qquad \begin{cases}(u_4,\,v_4)=(97,56);\; u_4=97,\; v_4=56;\\ c=97;\; a=55;\; b=57\\ S=55\cdot 57\cdot 97\cdot 112=34{,}058{,}640.\end{cases}$$

For RPT (97,55) using (3.1*.2) we have

$$S\,(97,\,55)=97\cdot 55(97^2-55^2)=$$
$$=55\cdot 57\cdot 97\cdot 112\;=S\;\text{in}\;(3.3^*.6).$$

3.4*. A solution of (3.1*. 4) $a^2 - a\,b + b^2 = c^2$.

In this section we state more triples of triangles RPT having the same area. We shall first prove .

Let

$$(3.4^*.1)\qquad \begin{cases}a^2-ab+b^2=c^2;\; a,\,b,\,c\in N-\{0\};\; then\\ b>c>a>0;\;(a,b,c)=1\end{cases}$$

In (3.4*. 1) we set arbitrary $b > a$; then we have

$$(3.4^*.2)\qquad \begin{cases}b^2-c^2=a(b-a)>0;\; b>c;\\ c^2-a^2=b(b-a)>0;\; c>a;\end{cases}$$

Thus $b > c > a > 0$, as stated in (3.4*. 1). We shall now prove

Theorem 3.4*.1.

Let

$$c^2=a^2-a\,b+b^2,\; b>c>a>0.$$

Then the three RPT-s formed by (b, c), (c, a), $(c, b - a)$ have the same area S, viz.

$$(3.4^*.\,3)\qquad S=a\,b\,c(b-a)$$

and there are infinitely many such triples RPT.

Proof:

If we set in (3.1*. 4) a $\to$ $-a$, we obtain from equation (3.1*.3), the equation (3.4*.1). But this is only an algebraic formality, since the RPT formed by $(c, -a)$ makes sense, though from $S = a\,b\,c(a + b)$ we obtain (3.4*.4) by substituting $-a$ for a. We also have

(3.4*.4) $c^2 = (a - b)^2 + a\,b\,,\quad c > a - b.$

Now

$$D_1 = \mathrm{RPT}(b, c);\ \ S(b, c) = \frac{1}{2}\cdot 2\,b\,c(b^2 - c^2) = \frac{1}{2}\,x\,y$$

$$= \frac{1}{2}\cdot 2\,b\,c\,a(b - a) = a\,b\,c(b - a);$$

$$D_2 = \mathrm{RPT}(c, a);\ \ S(c, a) = \frac{1}{2}\cdot 2\,c\,a(c^2 - a^2) = \frac{1}{2}\,x\,y$$

$$= \frac{1}{2}\cdot 2\,c\,a\,b(b - a) = a\,b\,c(b - a);$$

$$D_3 = \mathrm{RPT}(c, b\text{-}a);\ S(c, b\text{-}a) = \frac{1}{2}\cdot 2\,c(b - a)\,[c^2\text{-}(b - a)^2] = \frac{1}{2}x\,y$$

$$= c(b - a)ab = a\,b\,c(b - a);$$

$$S = S(b, c) = S(c, a) = S(c,\ b - a) = a\,b\,c(b - a).$$

We still have to prove that (3.4*.1) has infinitely many solutions. We shall give two methods to find these solutions, though these may not be all the infinitely many solutions of (3.4*.1). This was done by Hasse [33] with algebraic number theory which we shall avoid here, giving simple methods to solve (3.4*.1) in explicit form.

In (3.4*.1) we set

(3.4*.5) $b = a + v,\ \ v = b - a > 0;$

we obtain, setting $a = b - v$,

$$(b - v)^2 - b(b - v) + b^2 = c^2$$

$$b^2 - bv + v^2 - c^2 = 0$$

(3.4*.6) $v^2 - vb - (c^2 - b^2) = 0.$

From (3.4*.6) we obtain

$$v = \frac{1}{2}\left(b \pm \sqrt{b^2 + b(c^2 - b^2}\right),$$

(3.4*.7) $v = \dfrac{1}{2}\left(b \pm \sqrt{4c^2 - 3b^2}\right).$

We have, since setting $4\,c^2 - 3\,b^2 = x^2$ contradicts our condition $(a, b, c) = 1$

(3.4*.8) $4\,c^2 - 3\,b^2 = 1.$

Thus we have arrived at Pell's equation, setting

(3.4*.9) $2\,c = u_n\,,\ b = v_n.$

Thus un is even, and we must take $u_n = u_{2m+1}$ and from (3.2*.7*a*) we obtain

$$(3.4^*.10)\qquad \begin{cases} b = v_{2m+1}\,;\ c = \dfrac{1}{2}\,u_{2m+1}\,; \\ a = b - v = b - \dfrac{1}{2}(b \pm 1) \end{cases}$$

Since we obtain two values for a, we may have obtained six RPT-s with equal area. We shall investigate this later. We first take $v = \frac{1}{2}(b+1)$

$$(3.4^*.11)\qquad \begin{cases} a = \dfrac{1}{2}(b-1) = \dfrac{1}{2}(v_{2m+1} - 1); \\ b = v_{2m+1}\,;\ c = \dfrac{1}{2}\,u_{2m+1}\,; \\ b - a = \dfrac{1}{2}(v_{2m+1} + 1). \end{cases}$$

Hence, from (3.4*.3), or forming

$D_1 = \mathrm{RPT}(b, c),\ D_2 = \mathrm{RPT}(c, a),\ D_3 = \mathrm{RPT}(c, b\text{-}a)$

$$S = \frac{1}{2}(v_{2m+1} - 1)\cdot v_{2m+1}\cdot \frac{1}{2}\,u_{2m+1}\cdot \frac{1}{2}(v_{2m+1} + 1)$$

$$(3.4^*.12)\qquad S = \frac{1}{8}\,u_{2m+1}\cdot v_{2m+1}\,(v_{2m+1}^2 - 1).$$

We take for example $m = 1$,

$(u_3, v_3) = (26, 15)$

$u_{2m+1} = 26;\quad v_{2m+1} = 15$

$$S = \frac{1}{8}\cdot 26\cdot 15(15^2 - 1) = 10,920.$$

The reader should note that since

$$v_{2m+1} = 2S + 1,\ v_{2m+1}^2\text{-}1 \equiv 0 \pmod 8).$$

Hence in (3.4*.12) S is an integer. We now take $v = \frac{1}{2}(b - 1)$ and obtaine from (3.4*.10)

$$(3.4^*.13) \qquad \begin{cases} b = v_{2m+1}\,;\ c = \frac{1}{2}\,u_{2m+1}\,; \\ a = b - v = \frac{1}{2}(b+1) = \frac{1}{2}(v_{2m+1}+1) \\ b - a = \frac{1}{2}(v_{2m+1}-1). \end{cases}$$

and from D_4 = RPT(b, c), D_5 = RPT(c, a), D_6 = RPT(c, b-a). Previously, we had for $v = \frac{1}{2}(b+1)$,

$$a = \frac{1}{2}(v_{2m}-1);\ b = v_{2m+1}\,;\ c = \frac{1}{2}u_{2m+1}$$

$$b - a = \frac{1}{2}(v_{2m}+1).$$

Comparing (3.4*.11) with (3.4*.13) we see that

$$D_2 = D_6$$

$$D_3 = D_5$$

$$D_1 = D_4$$

And thus get the same triple of RTS-s in both cases.

3.5*. Second solution of $c^2 = a^2 - ab + b^2$.

We give a second method of finding infinitely many solutions of (3.4*.1). We set here

$$(3.5^*.1) \qquad a = \frac{1}{2}(y+1);\ \ b = y - 1;\ \ y \geq 2;\ \ y \in \mathbb{N}.$$

Substituting these values in (3.4*.1), we obtain

$$\left[\frac{1}{2}(y+1)\right]^2 - \frac{1}{2}(y^2-1) + (y-1)^2 = c^2$$

$$y^2 + 2y + 1 - 2y^2 + 2 + 4y^2 - 8y + 4 = 4c^2$$

$$3y^2 - 6y + 7 = 4c^2$$

$$3(y-1)^2 + 4 = 4c^2$$

(3.5*.2) $$4c^2 - 3(y-1)^2 = 4.$$

Since from (3.2*.7) $y - 1$ has to be even we can cancel (3.5*.2) by 4 and we obtain

(3.5*.3) $$y - 1 = 2v_{2m}, \quad y = 2v_{2m+1};$$

and with $c = u_{2m}$ we obtain

$$u_{2m}^2 - 3v_{2m}^2 = 1$$

$$a = \frac{1}{2}(2v_{2m} + 2),$$

(3.5*.4) $$a = v_{2m} + 1; \quad b = 2v_{2m}, \quad c = u_{2m}$$

$$S = a\,b\,c(b-a) = (v_{2m}+1)\,2\,v_{2m} \cdot u_{2m}(v_{2m} - 1)$$

(3.5*.5) $$S = 2\,u_{2m}\,v_{2m}\left(v_{2m}^2 - 1\right).$$

Comparing (3.5*.5) with (3.4*.12) we may think about the different expressions for area S; though one is expressed by u_{2m+1}, v_{2m+1}, the other by u_{2m}, v_{2m}, it is so because we are dealing here with different areas, depending on the values of a, b, c, which are different in each case. As an example for (3.5*.5) we choose $m = 2$,

$$(u_4, v_4) = (97, 56),$$
$$S = 2 \cdot 97 \cdot 56 \cdot (56^2 - 1) = 34,058,640.$$

If we set

$$a = 2m - 1, \quad b = 2v_{2m}, \quad c = u_n, \quad b - a = v_{2m} + 1,$$

(3.5*.6) $$u_{2m}^2 - 3v_{2m}^2 = 1$$

then $$a^2 - ab + b^2 = c^2, \quad S = 2\,u_{2m}\,v_{2m}\left(v_{2m}^2 - 1\right).$$

But we obtain nothing new since RPT(c, a), RPT(c, b - a) are interchanged with RPT(c, b - a), RPT(c, a).

The author is asking whether two succesive Fibonacci nambers could be solutions a, b of $a^2 \pm ab + b^2 = c^2$. The Fibonacci sequence, see [30] with $F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$, $n = 1, 2, \ldots$ goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … . Now $(F_4, F_5) = (3, 5)$ is a solution of the Diophantine example

$$3^2 + 3 \cdot 5 + 5^2 = 7^2,$$

and $(F_5, F_6) = (5, 8)$ is a solution of

$$5^2 - 5 \cdot 8 + 8^2 = 7^2.$$

Here $S = abc(b - a) = 5 \cdot 8 \cdot 7 \cdot 3 = 840$ whether there are more pairs of adjacent Fibinacci numbers serving as solutions of $a^2 - ab + b^2 = c^2$ could not be decided generally here.

3.6*. Perimeters and areas

As known, P, half the perimeter of a RPT is a divisor of its area S, since

(3.6*.1) $\quad P = u(u + v), \quad S = u\,v(u^2 - v^2)$

$$S \div P = v(u - v)$$

Where (u, v) forms the RPT. The quaestion arises whether there are other connections between these two elements. Here we could only prove

Theorem 3.6*.1.

Let $a^2 - ab + b^2 = c^2$, $b > c > a > 0$, $a, b, c \in \mathbb{N}-\{0\}$. Then the sum of the three perimeters formd by a, b, c, viz. RPT(b, c), RPT(c, a), RPT$(c, b\text{-}a)$ is the sum of two aquares.

Proof: We have

$D_1 = \mathrm{RPT}(b, c)$: $\quad 2P_1 = x_1 + y_1 + z_1 = b^2 - c^2 + 2\,bc + b^2 + c^2$

$$= 2\,b(b + c)\,;$$

$D_2 = \mathrm{RPT}(c, a)$: $\quad 2P_2 = x_2 + y_2 + z_2 = c^2 - a^2 + 2\,ca + c^2 + a^2$

$$= 2\,c(c + a)\,;$$

$D_3 = \mathrm{RPT}(c, b - a)$: $\quad 2P_3 = x_3 + y_3 + z_3 = c^2\text{-}(b\text{-}a)^2\text{+}2c(b\text{-}a)\text{+}(b\text{-}a)^2 + c^2$

$$= 2\,c^2 + 2\,c(b - a) = 2\,c(c + b - a).$$

$$P_1 + P_2 + P_3 = b(b + c) + c(c + a) + c(c + b) - a\,c$$
$$= 2\,c^2 + 2b\,c + b^2 = (b + c)^2 + c^2$$

3.7*. Entirely new RPT-s.

In a very little known paper Hillyer [35] has giving a most surprising infinity of triples of Rational Pythagorean tiangle in an explicit form, having equal areas. These are formed by

$$(3.7^*.1) \quad \begin{cases} D_1 = RPT(u, v) = RPT(a^2 + ab + b^2,\ b^2 - a^2) \\ D_2 = RPT(u, v) = RPT(a^2 + ab + b^2,\ 2ab + a^2) \\ D_3 = RPT(u, v) = RPT(2ab + b^2,\ a^2 + ab + b^2) \\ b > a > 0;\ b, a \in Q^+ \end{cases}$$

As we see, Hillyer was not concerned about RPT-s; his Pythagorean triangle had just to have rational sides. We shall operate with RPT-s only and set

$$(3.7^*.2) \qquad a, b \in \mathbb{N} - \{0\}; \quad b > a > 0.$$

We first investigate, whether the condition

$$(3.7^*.3) \qquad u > v > 0$$

is fulfilled for D_1, D_2, D_3. We have

$$D_1 : u - v = a^2 + ab + b^2 - (b^2 - a^2)$$
$$= 2b^2 + ab > 0;\ u > v > 0.$$

$$D_2 : u - v = a^2 + ab + b^2 - (2ab + a^2)$$
$$= b^2 - ab = b(b - a) > 0,$$

since $b > a$ hence $u > v > 0$

$$D_3 : u - v = 2ab + b^2 - (a^2 + ab + b^2)$$
$$= ab - a^2 = a(b - a) > 0,$$

since $b - a > 0$, $u > v > 0$.

We shall now find the areas of D_1, D_2, D_3, and have

$$D_1 : S_1 = (a^2 + ab + b^2)(b^2 - a^2)[(a^2 + ab + b^2)^2 - (b^2 - a^2)^2]$$
$$= (a^2 + ab + b^2)(b^2 - a^2)(a^2 + ab + b^2 - a^2)$$
$$\cdot (a^2 + ab + b^2 - b^2 + a^2)$$
$$= (a^2 + ab + b^2)(b^2 - a^2)(ab + 2b^2)(ab + 2a^2).$$
$$= (a^2 - ab + b^2)(b^2 - a^2)\,a(2a - b)(2b - a)$$

$$(3.7^*.4) \qquad S_1 = ab(a^2 + ab + b^2)(b^2 - a^2)(a + 2b)(b + 2a).$$

$$D_2 : S_2 = (a^2 + ab + b^2)(2ab + a^2)[(a^2 + ab + b^2) - (2ab + a^2)^2]$$
$$= (a^2 + ab + b^2)\,a(a + 2b)(a^2 + ab + b^2 + 2ab + a^2)\cdot$$
$$\cdot (a^2 + ab + b^2 - 2ab - a^2)$$
$$= (a^2 + ab + b^2)\,a(a + 2b)(2a^2 + 3ab + b^2)(b^2 - ab).$$

Now

$$2a^2 + 3ab + b^2 = (2a + b)\,(a + b),$$

hence

$$(3.7^*.\,5) \qquad S_2 = ab(a^2 + ab + b^2)\,(b^2 - a^2)\,(a + 2b)\,(b + 2a)$$

$$D_3 : S_3 = (2ab + b^2)\,(a^2 + ab + b^2)\,[(2ab + b^2)^2 - (a^2 + ab + b^2)^2]$$
$$= b(2a + b)\,(a^2 + ab + b^2)\,(2b^2 + 3ab + b^2)\,(ab - a^2).$$

Now

$$2b^2 + 3ab + b^2 = (2b - a)\,(b + a).$$

Hence

$$(3.7^*.6) \qquad S_3 = ab(a^2 + ab + b^2)\,(b^2 - a^2)\,(a + 2b)\,(b + 2a).$$

Thus $S_1 = S_2 = S_3 = \Sigma$,

$$(3.7^*.7) \qquad \Sigma = ab(a^2 + ab + b^2)\,(b^2 - a^2)\,(a + 2b)\,(b + 2a).$$

We now return to the equation of Diophantus from (3.1*.3)

$$(3.1^*.3) \qquad \begin{cases} a^2 + ab + b^2 = c^2 \\ a + b > c > b > a > 0 \\ a,\ b,\ c \in N - \{0\}. \end{cases}$$

We found that the three triples RPT(*c*, *b*), RPT(*c*, *a*), RPT(*a*+*b*, *c*) have equal areas, viz.

$$S = a\,b\,c\,(b + a)$$

If in (3.7*.7) we demand that $a^2 + ab + b^2 = c^2$, solvable in natural number $a + b > c > a > a > 0$ we obtain

$$(3.7^*.8) \qquad \Sigma = abc^2\,(b^2 - a^2)\,(a + 2b)\,(b + 2a),$$

and from (3.7*.8), and $S = a\,b\,c\,(a + b)$,

$$(3.7^*.9) \qquad \Sigma \div S = c\,(b - a)\,(a + 2b)\,(b + 2a).$$

Now the quotent $\Sigma \div S$ is a natural number, and many authors have asked and solved the question of the ration of the areas of two RPT-s.

3.8*. The main result

We now form three RPT-s, having equal areas. They are entirely new and unknown. We investigate

$$(3.8^*.1)\quad \begin{cases} D_1 = RPT(u, v) = RPT(a^2 - ab + b^2,\ b^2 - a^2) \\ D_2 = RPT(u, v) = RPT(a^2 - ab + b^2,\ 2ab + b^2) \\ D_3 = RPT(u, v) = RPT(2ab - a^2,\ a^2 - ab + b^2) \\ 2a > b > a > 0;\ b, a \in Q^+ \end{cases}$$

We first check for D_1, D_2, D_3,

$$u > v > 0,\ u - v > 0.$$

The reader should note the condition $2a > b$.

Later when we shall operate with RPT-s and the equation

$$a^2 - ab + b^2 = c^2\ ,$$

we shall see that solutions of this equation are possible with $2a > b$.

We have, $(u, v) \in Q^+$.

$$D_1 = u - v = a^2 - ab + b^2 - (b^2 - a^2)$$
$$= 2a^2 - ab = a(2a - b) > 0.$$
$$v = b^2 - a^2 > 0.$$

$$D_2 : u - v = a^2 - ab + b^2 - (2ab - b^2)$$
$$= a^2 - 3\ ab + 2b^2$$
$$= (2b - a)\ (b - a) > 0$$
$$v\ = 2ab - b^2 = b(2a - b) > 0.$$

$$D_3 : u - v = 2\ \mathrm{ab} - a^2 - (\mathrm{a}^2 - ab + b^2) = 3ab - 2a^2 - b^2$$
$$= (2a - b)\ (b - a) > 0$$
$$u\ = 2ab - a^2 = a(2b - a) > 0.$$
$$v\ = a^2 - ab + b^2 = (a - b)^2 + ab > 0.$$

We shall now find the areas formed by D_1, D_2, D_3, and have

$$D_1 : S_1 = (a^2 - ab + b^2)\ (b^2 - a^2)\ [(a^2 - ab + b^2)^2 - (b^2 - a^2)]$$
$$= (a^2 - ab + b^2)\ (b^2 - a^2)\ (a^2 - ab + b^2 - b^2 + a^2)^2 - (a^2 - ab + b^2 + b^2 - a^2)$$
$$= (a^2 - ab + b^2)\ (b^2 - a^2)\ (2a^2 - ab)\ (2b^2 - ab)$$

$= (a^2 - ab + b^2)(b^2 - a^2)\, a(2a - b)(2b - a)$

(3.8*.2) $\qquad S_1 = ab(a^2 - ab + b^2)(b^2 - a^2)(2a - b)(2b - a)$

$D_2 : S_2 = (a^2 - ab + b^2)(2ab - b^2)[(a^2 - ab - b^2) - (2ab - a)^2]$

$= (a^2 - ab + b^2)\, b(2a - b)(a^2 - ab + b^2 - 2ab + b^2)(ab + a^2)$

$= (a^2 - ab + b^2)\, b(2a - b)(a^2 - 3ab + 2b^2)\, a(b + a)$

$= (a^2 - ab + b^2)\, b(2a - b)(2b - a)(b - a)\, a(b + a)$

$= ab(a^2 - ab + b^2)(b^2 - a^2)(2a - b)(2b - a)$

(3.8*.3) $\qquad S_2 = ab(a^2 - ab + b^2)(b^2 - a^2)(2a - b)(2b - a)$

$D_3 : S_3 = (2ab - a^2)(a^2 - ab + b^2)[(2ab - a^2)^2 - (a^2 - ab + b^2)^2]$

$= (2ab - a^2)(a^2 - ab + b^2)(2ab - a^2 - a^2 + ab - a^2)(2ab - a^2 + a^2 - ab + b^2)$

$= a(2b - a)(a^2 - ab + b^2)(3ab - 2a^2 - b^2)(ab + b^2)$

$= ab(a^2 - ab + b^2)(2b - a)(2a - b)(b - a)(a + b)$

$= ab(a^2 - ab + b^2)(b^2 - a^2)(2a - b)(2b - a)$

(3.8*.4) $\qquad S_3 = ab(a^2 - ab + b^2)(b^2 - a^2)(2a - b)(2b - a)$.

Thus we have obtained the desired result

$$S_1 = S_2 = S_3 = \Sigma'$$

We now return to the equation

(3.4*.1) $$\begin{cases} a^2 - ab + b^2 = c^2,\ b > a > 0 \\ b > c > a > 0;\ b, c, a \in N - \{0\} \end{cases}$$

and recal from (3.5*.4) that there is a solution with

(3.8*.5) $$a = \frac{b}{2} + 1;\ 2a > b$$

as we needed. With equation (3.4*.1), Σ' takes the form

(3.8*.6) $$\begin{cases} \Sigma' = abc^2(b^2 - a^2)(2a - b)(2b - a) \\ 2a > b > c > a;\ a, b, c \in N - \{0\}. \end{cases}$$

Now with $a^2 - ab + b^2 = c^2$, and (3.4*.3), viz.

$$S = a\, b\, c\, (b - a)$$

and obtain thus the quotient

(3.8*.7) $\Sigma \div S = c\,(b + a)\,(2a - b)\,(2b - a).$

As an exemple we shall take

$$a^2 - ab + b^2 = c^2$$

$$(a, b, c) = (8, 15, 13).$$

Here $2a = 16 > 15 = b$; we obtain

$$\Sigma` = 8 \cdot 15 \cdot 169 \cdot 161 \cdot 1 \cdot 22,$$

$$\Sigma` = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 23 = 71,831,760.$$

$$S = a\,b\,c\,(b - a) = 8 \cdot 15 \cdot 13 \cdot 7$$
$$= 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 10,920.$$

$$\Sigma` \div S = 2 \cdot 11 \cdot 13 \cdot 23 = 6,578$$

$$71,831,760 \div 10,920 = 6,578.$$

Section 3.3. Units form (BGEA) play an important role in solving (EFLT).

In this section we shall give another justification of Hilbert's work relating the existence of integer solution for $x^2 + y^2 = z^2$ and the unrestricted periodicity of the (EA).

In section (1.1.2) we mentioned that the unrestricted periodicity of the (EA) is a very important property and in the quadratic case it inables us to solve the Euler – Pell's equations $x^2 - m\,y^2 = \pm 1$ or ± 4, where m is a square free natural number, and to find the fundamental unit in the quadratic field $Q\left(\sqrt{m}\right)$.

For the quadratic extensions of Q, the multiplicative group of units was completely determined when the above equations were completely solved by simple continued fractions or from (EA) unrestricted periodicity (2 – ELT).

This group of units was formerly refered to as the Galois' group of units in the algebraic field. In [37] pag. 244 a Lemma is proved which state that the Galois group of the cyclotomic field of order n over F of characteristic O is abelian by showing that it is isomorphic to the abelian group of units of the ring $Z/(n)$. Many mathematicians seem to object to this nomenclature and therefore we can drop Galois and we stay with the old name of the multiplicative group of units in the algebraic number field.

Thus the calculation of units and Diriclet's problem are related with the solvability of the Diophantine equations.

In the previous section we solved explicitly the quadratic Diophantine equation of the form $a^2 \pm ab + b^2 = c^2$.

Hasse gave the totality of solutions of these equations in parameter from anf they are known as Hasse's equations. The author was able to give explicite solutions for these equations and with a proper linear transformation, they can be reduced to a simple Euler – Pell's equation which can be solved explicity.This is posible in quadratics since (EA) is unrestricted periodic.

In Sections (3.3), (4.3) and (4.4) of [23] we used units derived from (BGEA) to solve more Diophantine equations of higher degrees. With this example we give another justification for using the periodicity of (BGEA) in giving solutions of some higher degree Diophantine equations.

Section 3.4. (BGEA) restricted periodicity and the Euclidean solution of the original Fermat's Last Theorem stated in Euclidean Terms (EFLT).

Theorem 3.4.1.

The Euclidean Fermat's Last Theorem (EFLT).

There do not exist positive integers x, y, z, n such that

$$x^n + y^n = z^n ; \text{ if } n \geq 3.$$

Theorem 3.4.2.
(BGEA) restricted periodic when $n \geq 3$ implies (ELFT).

Proof:
It is known that in quadratics $x^2 + y^2 = z^2$ has integer solutions and this is an immediate consequence of the fact that (EA) is periodic. We know that Euclidean Algorithm (EA), Jacobi Algerithm (JA), Perron Algorithm (PA), Jacobi – Perron Algorithm (JPA) and Hasse – Bernstein Algorithm (HBA) developed only real numbers of the form $w = \sqrt[n]{D^n + d}$ and the (HBA) was

developed only real numbers of the form $w = \sqrt[n]{D^n + d}$ and the (HBA) was the closest algorithm over the reals to the General Euclidean Algorithm (BGEA). (HBA) of

$$w = \sqrt[n]{D^n + d}\ , \mathrm{D} \in \mathbb{N},\ d \in \mathbb{Z},\ n \geq 3$$

is periodic when

$$d > 0,\ \mathrm{D} \geq (n\text{-}2)\ d,\ d\ /\ \mathrm{D}$$

and when

$$d < 0,\ \mathrm{D} \geq 2(n\text{-}1)\ d,\ d\ /\ \mathrm{D}.$$

Baica developed (HBA) for the first time over the complex numbers and that eliminated the restrictions on D and only *d*/D remains in order to prove periodicity.

We named this Algorithm over the complex number (BGEA) since no other larger set of numbers in range exists having the complex numbers as a subset as $\mathrm{R} \subset \mathrm{C}$. Then, we realized that (BGEA) is an explicit form of Hilbert's demanded universal algorithm from whose periodicity all of the open problems in *n*-dimensions could be solved, which have been already solved in quadratics (n = 2) from the periodicity of (EA) under the form of continued fractions. Logicians proved that Hilbert's hoped for periodic Algorithm does not exist and this is known as Hilbert's 10-th Problem. We proved exactly the same result, providing Mathematics with an explicit (BGEA) which is periodic for any higher degree algebraic number $w = \sqrt[n]{D^n + d}$ if *d*/D. Putting those two together it is true that if d does not divide D, (BGEA) is not periodic, since otherwise it will contradict Hilbert's 10-th Problem proved by logic. (BGEA) is of the same cut or prototype as (EA) for starting vectors whose components are real numbers which are higher degree irrationals.

That (BGEA) is the (GEA) is not in doubt, under its much less powerful form it was used by great mathematicians to approach similar open questions in *n*-dimensions which were proved in quadratics from the periodicity of (EA). Therefore, if $n \geq 3$ there do not exists positive integers x, y, z such that

$$x^n + y^n = z^n$$

since (BGEA) is not always periodic for $n \geq 3$, and there exists positive integers

x, y, z such that $x^2 + y^2 = z^2$ since for $n = 2$ (BGEA) for reals becomes (EA) and (EA) is periodic for any quadratic irrational. If (BGEA) would be periodic for any n, then

$$x^n + y^n = z^n$$

would have integer solutions but this will contradict Hilbert's 10-th Problem. Therefore (BGEA) not periodic for $n \geq 3$ if d does not divide D implies Fermat's Last Theorem.

Note:

(EA) is $n = 2$ for (BGEA). Because of its periodicity many problems have been solved in quadratics ($n = 2$).

The degree of the irrational (quadratic) is related with the degree of the equation $x^2 + y^2 = z^2$ having integer solutions because any quadratic irrational w will make (EA) periodic. Like wise the degree n of the irrational w in (BGEA) is related with the degree n of $x^n + y^n = z^n$ (FLT). The quantity under the n-th degree radical $D^n + d$ is related with the proof of the periodicity of (BGEA) for that corresponding $w = \sqrt[n]{k}$ where k can be written always as $k = D^n + d$. The condition d/D is required to prove (BGEA) of w to be periodic as it's one of the many other conditions to prove periodic (HBA) for the same w. No hundreds of pages of proof are needed. The Euclidean Model is explained by the History of Mathematics and is well known by most mathematicians; it is not necessary to explain it from scratch. Also, (BGEA)[1] is published. All of those great mathematicians from the History of Mathematics, starting with Euclid, helped me to prove (EFLT) in the Euclidean Model. (BGEA) answers all of the other open questions in higher degrees up to its periodicity; d/D is required to prove periodicity and this condition cannot be eliminated. This is the key to proving (EFLT). The proof of (EFLT) is the work of Euclid, Jacobi, Perron, Gauss, Euler, Hermite, Hilbert, Dirichlet, Hasse, Bernstein and Baica put together. All of those great mathematicians before me ultimately were hoping to solve

(EFLT) and historically they paved the way for me to finish the final step in its proof. The solution of (EFLT) is the evolutionary development of the algorithms of Jacobi, Perron, Hasse-Bernstein, and Baica.

Note:

Jacobi and Perron were interested in developing an Algorithm whose periodicity would assist in the solution of Hermite's Problem. Hasse-Bernstein were interested in solveng Dirichlet's Problem. The application of the Theory of Units cannot be sufficiently prized. Gauss himself used it to prove the truth of Fermat's conjucture in the cubic case by using units in the quadratic algebraic field $Q\left(\sqrt{-3}\right)$ and Kummer in his effort to solve (EFLT) had recourse to the units of cyclotomic fields. London and Finkalstein [39] have written a whole book yielding information through the theory of units about the famous Merdell equation.

(BGEA) now solves Dirichlet`s problem completely, satisfying Hilbert`s demand, and the application of units will demeonstrate its dominating strength.

In conclusion (BGEA) is a very powerful algorithm when it becomes periodic. The proof of (EFLT) is the conclusion of the results in all author's papers over the years.

The (BGEA) will dominate mathematics for higher dimensional fields in the years to come, exactly as (EA) $n = 2$ in (BGEA) dominated mathematics for quadratic fields for so many years in the past.

The applications of (BGEA) do not stop here. In many other published papers we have extended the application of the periodicity of (BGEA) to solutions of very complicated diophantine equations, we have developed very complicated combinatorial identities and recently, we used it to find the sums of some infinite series. For the first time we emphasized the importance of the periodicity of an algorithm as a tool to prove something in Mathematics.

Section 3.5. More explanations about Baica's proof of the Euclidean Fermats's Last Theorem

In this section the author will answer some questions raised at some various profesional conferences and meetings when she presented her proof [13] of Fermat's Last Theorem (EFLT).

Question 3.5.1.

At the International Conference on Analytic Number Theory in Allerton Park, at the University of Illinois in Urbana-Champaign, May 16-20, 1995, the author was asked this question.

"Do you mean that if we come up with another new algorithm this time unrestricted periodic, than (EFLT) is false?"

Answer to the question

The answer is NO. In order for that new algorithm to make (EFLT) false this new algorithm must be The General Euclidean Algorithm.

This new General Euclidean Algorithm has to solve from its unrestricted periodicity all of the problems in Chapter 2 of [13] as (EA) solves all of the problems in Chapter 1 of [13] from its unrestricted periodicity.

(BGEA) solves all the problems in Chapter 2 of [13] up to its restricted periodicity. (BGEA) is the General Euclidean Algorithm since it is of the same cut or prototype as (EA) which is $n = 2$ in (BGEA), when Jacobi ($n = 3$ in (BGEA)) and Perron (any n in (BGEA)) first originated it.

In Chapter 3 of [13] the author identified all her publications in which she proved up to (BGEA) restricted periodicity all of those open questions in n - dimension (Chapter 2 of [13]).

In [13] the author proves explicitly Hilbert's 10-th Problem and with it she showed that (BGEA) is the general Euclidean Algorithm and this is the key in solving (EFLT) as it is the key in solving completely now all of the problems in Chapter 2 of [13] from its restricted periodicity.

Question 3.5.2.

At the Conference on Number Theory and Fermat's Last Theorem at Boston University on August 9-18, 1995 the author was asked this question

"How do you relate the restricted periodicity of your (BGEA) with the degree n of the equation

$$x^n + y^n = z^n$$

in the (EFLT) ?"

Answer to the question

Every $w = \sqrt{k}$ (quadratic irrational) makes (EA) unrestricted periodic. This is the Euler-Lagrange if and only if Theorem. Therefore (EA) is periodic unrestricted and because of its unrestricted periodicity many problems in quadratics are solved (Chapter 1 in [13]) and the same problems in n-dimensions were still open (Chapter 2 in [13]).

All of those open question for $n > 2$ caused Hilbert to ask for the invention of a universal algorithm of dimension n as powerfull as (EA) for quadratics in order to solve all of the problems in higher ($n > 2$) dimensions (Chapter 2 in [13]) from the priodicity of this universal algorithm. This is known as Hilbert's 10-th Problem.

Since many geat mathematicians before, including Hilbert, realted quadratics with the periodicity of (EA) and since it is known that

$$x^2 + y^2 = z^2$$

has integer solutions as an immediate consequence of (EA) being always periodic, we can likewise perform the generalization for any n.

In [1, 13], we proved that only some

$$w = \sqrt[n]{k} = \sqrt[n]{D^n + d} \quad \text{when } d / \mathrm{D}$$

n-th degree irrationals makes (BGEA) periodic. This is one direction in the proof of the periodicity of (BGEA).

The other direction proof of the periodicity of (BGEA) is exactly the same as given by Perron [40], where he proved that his algorithm is periodic if w is an algebraic number or any n -th degree irrational ($\mathrm{Q} = \sqrt[n]{k}$).

In [13] we proved that if d does not divide D, then (BGEA) is not periodic if $n \geq 3$, since otherwise (BGEA) would become Hilbert`s hoped for Universal Algorithm which would contradict the solution of Hilbert`s 10-th Problem.

In conclusion the periodicity of (EA) was connected with the degree $n = 2$ of the irratioal which makes (EA) unrestricted periodic and further with the existence of integer solutions

$$x^n + y^n = z^n \quad \text{when } n = 2.$$

Because some n-th degree irrationals make (BGEA) periodic the dimension of (BGEA) is n, and for $n = 2$, (BGEA) becomes (EA), similary we connect the dimension n of (BGEA) with the degree n in the equation

$$x^n + y^n = z^n$$

for $n \geq 3$, as the dimension $n = 2$ of (EA) in (BGEA) was connected with the dimension $n = 2$ in the equation

$$x^n + y^n = z^n$$

by many other mathematicians in the History of Mathematics before.

Question 3.5.3.

This question came in a letter received from Georg – August – Universitat in Gottingen, Germany dated 8-31-95. I received this comment:

"Finally let me comment on your paper. The crucial point, as it seems to me and also Prof. P., whom I asked, is that you want to generalize from quadratic fields and the Euclidean Algorithm to higher degrees and an extension of the algorithm. But you never give a detailed proof how periodicity of the algorithm and solvability of Fermat's Equation are connected. Therefore it remains very doubtful if such a connection exists. In many cases, as for example the Theory of Complex Functions of one several variables, results and proofs cannot be generalized to "higher degrees" and even if there is an analogy of prerequisites and wanted results, it might be impossible to generalize the proof. This is the reason, why in our opinion your proof lacks completeness and must be worked out".

In another words this question can be rephrased as:

"What gives you permission to generalize from the quadratics and the unrestricted periodicity of (EA) to the *n*-dimension and the restricted periodicity of (BGEA)" ?

This question is legitimate because the generalization may not be always possible.

Answer to the question

This is correct. The (BGEA) is crucial. I argue that the Algorithm developed in the paper [1] and now called Baica's Generalized Euclidean Algorithm (BGEA) is the tool that makes generalization possible.

The uses of a Generalized Algorithm and its periodicity were suggested by Hilbert. What I have done is to extend the work of Hasse and Bernstein [24] to obtain tools that can be used to attack Fermat's Last Theorem, in Euclidean terms.

What it was said is that it can be seen how the earlier paper that I wrote [1] and my paper on (EFLT) [13] are connected.

The existence of a generalized algorithm (BGEA) allows an induction on the degrees of the Fermat's equations. In each case for
$n \geq 3$, the algorithm (BGEA) is not periodic.

In Euclidean the principle of induction never fails to give the generalization.

The corresponding algebra of the Euclidean Geometry, in this case, is the Algebraic Number Theory, and that is a Peano-Algebra in which induction provides the generalization.

Fermat was thinking in the same direction of using induction for $n \geq 3$ but at that time he did not have the tool (BGEA) to legitimize his use of induction on the dimension n of the (BGEA) algorithm in his proof.

Question 3.5.4.

This question came in a letter received from Annali di Matematica Pura ed Applicata, Firenze, Italy on March 4, 1995.

"Let $w = \sqrt[n]{D^n + d}$ with D, d positive integers, $n \geq 3$ and d not a divisor of D.

It is possible to conclude that in such a case (BGEA) is never periodic ?" If yes, how can this be proved ?

Answer to the question.

This proof was given in [13]. It is true that all the proofs for periodicity of (HBA) [24], and BGEA [1], considered d / D, and this is only necessary condition to prove it periodic.

In [13] the author solved explicitly Hilbert's 10-th Problem and putting these two together that will make d/D an if and only if condition for the proof of the periodicity of (BGEA).

All of the proofs in Chapter 2 of [13] from the restricted periodicity of (BGEA) are now if and only if conditions.

From logic the negation of an if and only if condition is again an if and only if condition and therefore it is true that if d does not divide D then BGEA is not periodic.

In conclusion we solved more problems. Some of them include Hermite's Problem, Dirichlet's Problem, Hilbert's 10-th Problem and Galois' Theory of Polynomials Problem, not as controversial, but as difficult as, or more difficult than Euclidean Fermat's Last Theorem, using as the tool this restricted periodicity of the (BGEA).

Question 3.5.5.

Some mathematcians said that (BGEA) is not the (GEA) since the original Fibonacci Numbers can be derived from the periodic expansion by the (EA) of $\sqrt{5}$ or by a periodic continued fraction development of $\sqrt{5}$ only.

Answer to the question.

Many mathematicians wrongly refer to Baica`s Generalized Euclidean Algorithm (BGEA) as my General Continued Fractions Algorithm. It only happens that for $n = 2$ in (BGEA) which is (EA) it can be identified with the Continued Fractions because (2-ELT) proves the unrestricted periodicity on the (EA) using the Continued Fractions Algorithm (CFA). Jacobi and Perron used General Continued Fractions Algorithm (GCFA) but they could not prove the periodicity or the Restricted Periodicity of their (GCFA) Algorithm except for some numerical examples.

The transformation in the Continued Fractions Algorithm (CFA) is the greatest integer function. The (HBA) was the closest Algorithm to the (GEA) for reals and they did not use greatest integer function as their transformation, but instead, they use the evaluation function as their transformation [24]. Baica used the same transformation as (HBA) for the first time over the Complex Number Field and with this she proved an if and only if theorem for her (BGEA) restricted periodicity. I did not claim that my Algorithm (BGEA) is anything else excepted the Generalized Euclidean Algorithm (BGEA) and it is more than the (GCFA).

In [2] the author opened a new horizon for the desired generalization of the Fibonacci Numbers and used (BGEA) restricted periodicity to derive n-dimensional Fibinacci Numbers, and it first turn out that these n-dimensional Fibonacci Numbers are most useful for a good approximation of Algebraic Irrationals by Rational Integers. For $n = 2$ in the n-dimensional Fibonacci Numbers in [2] form (BGEA) we obtain the Original Fibonacci Numbers.

Question 3.5.6.

In [21] we claimed that the (BGEA) restricted periodicity proof gives the first algorithmic explicit solution to Hilbert`s 10-th Problem. There are complaints about that claim saying that there exists an explicit Polynomial Representation proof for Hilbert`s 10-th Problem.

Answer to the question.

My answer to this question is that this Explicit Polynomial representation is not an algorithmic explicit representation as (BGEA) is to comply with Hilbert`s demand for the invention of his hoped - for Algorithm to be unrestricted periodic.

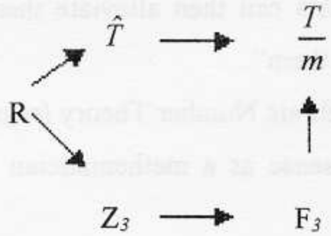
Hilbert demended the General Euclidean Algorithm (GEA) and the proof of its unrestricted periodicity, where instead we invented (BGEA) restricted periodic which does not contradict the logicians` solution of Hilbert`s 10-th Problem.

Thus this (BGEA) is the only algorithmic explicit solution of Hilbert`s 10-th Problem.

Section 3.6. A. Wiles attempt to prove the Elliptical Fermat`s Last Theorem (ELFLT).

We are not going to reproduce A. Wiles' work here, our intention is to underline the weakness of his proof in his attempt to prove (ELFLT).

Modularity is essential in the proof of (ELFLT). Using deformations, the constructed l-ADIC representation for l = 3, starting with the representation on the 3-division points, known to be congruent to a modular representation, leads to the following commutative diagram:



A. Wiles wanted to show that R is isomorphic to $\hat{T}$ since then the elliptic Galois representation becomes modular.

There is no information about R from the general principles of its construction explained in the proof. With special considerations in the proof $R = \hat{T}$ if and only if the order $\frac{p}{p^2}$ = the order of $\frac{Z_3}{\eta \cdot Z_3}$.

Let us denote the order $\left|\frac{p}{p^2}\right| = \left|\frac{Z_3}{\eta \cdot Z_3}\right|$

A.Wiles tried to establish equality by using Euler Systems given by

Kolyvagin. The equality could not be proved because Euler Systems of Higher Levels could not be constructed.

Thus, A. Wiles did not prove (ELFLT).

Section 3.7. G. Faltings finishes the proof of the Elliptical Fermat's Last Theorem (ELFLT).

The July 1995, Notices of the AMS published a translation from the german of G. Faltings' March 1995 Article [32] in which the author said at the beginning of the paper: "The proof of the conjecture mentioned in the title was finally completed in September of 1994. A. Wiles announced this result in the summer of 1993. However, there was a gap in his work.

"The paper of Taylor and Wiles does not close this gap but circumvents it."

This article is an adaptation of several talks that I have given on this topic and is by no means about my own work".

With this statement one of my concerns was justified. A. Wiles did not prove (ELFLT). It is G. Faltings who did it.

He said "The specialists can then alleviate their boredom by finding these mistakes and correcting them".

I am specialist in Algebraic Number Theory (regretably very few left in this field) but my common sense as a mathematician tells me to accept G. Falting's Proof of (ELFLT).

Faltings began where A. Wiles got stuck. He starts from the minimal level M and then reduces to it. The R_e and $\hat{T}_e$ are rings of power seried and they become equal at the limit. R is obtained from R_e and $\hat{T}$ from $\hat{T}_e$.

To prove the equality in (3.7.1)

$$(3.7.1) \qquad \left|\frac{p}{p^2}\right| = \left|\frac{Z_3}{\eta \cdot Z_3}\right| \quad \text{where } |\ | \text{ means order}$$

is to reduce the problem to the minimal case and he estimates how both sides of the inequality (3.1.2).

(3.7.2) $$\left|\frac{p}{p^2}\right| \geq \left|\frac{Z_3}{\eta \cdot Z_3}\right|$$

change as one proceeds from minimal level M to a higher level N.

An uper bound for the left hand side and a lower bound for the right hand side were found, and they coincided. This ends the proof of (ELFLT).

Section 3.8. Elliptical Fermat's Last Theorem (ELFLT) is equivalent to (not the same as) the Original Fermat's Last Theorem (EFLT) stated by Fermat in Euclidean Terms.

As we have said before among the many geometries constructed only one is the Euclidean Geometry; all of the others are Noneuclidean. The Geometries do not relate to each other, but they all relate to the topology. Because of this we can prove something in one geometry which may not be proved in another geometry, or it may mean something else in another geometry. The best that we can get is that these results are equivalent or isomorphic. Also, these results are transferred in the computing algebras of the corresponding geometries. To prove that two results are equivalent we have to provide the transformation which is a one to one correspondence function, to prove that the results are the same in these two different categories we need the Galois Connection which requires the analytical continuity. In other words we need to provide a one to one, onto analytical continuous function which is a Functor.

When we transfer a result from one Computing Algebra System to another Computing Algebra System we need this Functor.

So far in Faltings' proof of (ELFLT) only the transformation for equivalence was given since both proofs Baica's of (EFLT) and Faltings of (ELFLT) bring us to the solvability by radicals, and this is related with the multiplicative group of units in the corresponding fields. We explained the importance of finding the multiplicative group of units in solving (EFLT) before in Section 3.3. of this book.

The Functor was not yet provided to show that (ELFLT) is the same as (EFLT) and this may not exist.

Section 3.9. Conclusions.

This problem (EFLT) has baffled the best mathematicians for nearly 350 years. We, finaly, proved that the restricted periodicity of (BGEA) for $n \geq 3$ implies Fermat.

Fermat himself intended to prove (EFLT) by induction since the Classical Number Theory is a Peano-Algebra and there the unduction never fails to give the generalization. He first proved his conjecture to be true for $n = 4$ but when he wanted to do induction on n he could not use the degree of the equation. The induction must be done on the dimension n of the (BGEA) where $n = 2$ in (BGEA) is (EA), and the dimension is given by the degree of the irrational which makes (BGEA) restrictive periodic. The dimension of (BGEA) brings us to the degree of the equation using the soluability by radicals.

At that time Fermat did not have (BGEA) to perform induction on its dimension.

Thus we proved the (EFLT) by proving (BGEA) restrictive periodic if and only if (EFLT) original Fermat's Last Theorem.

As we see now the restricted periodicity of (BGEA) has everything to do with the Original Euclidean Fermat's Last Theorem.

The equation $x^n + y^n = z^n,\ \ n \geq 3$

does not resemble any of the Diophantine equations solved by the author in [4], using units in the Algebraic Number Fields, by the restricted periodicity of (BGEA), and this is the reason why (BGEA) restricted periodicity is equivalent to (EFLT).

Baica's proof of (EFLT) is the work of Euclid, Jacobi, Perron, Gauss, Euler, Lagrange, Hermite, Hilbert, Dirichlet, Hasse, Bernstein and Baica put together. All of these great mathematicians before me ultimately were attempting to solve (EFLT) and historically they paved the way for me to finish the final step in its proof. Baica's solution of (EFLT) is the evolutionary development of the algorithms of Jacobi, Perron, Hasse, Bernstein and Baica, and it is the only solution provided in Euclidean Terms so far. The tool is (BGEA) and we got the solution putting toghether all related work from the history of mathematics starting with Euclid up to our time.

As we see, it is not only the beginning Euclidean Algorithm (EA), and the end Baica's Generalized Euclidean Algorithm (BGEA), but there is so much else in the gap that separates them by more than 2000 years. Our result puts together the work of great mathematicians during the entire History of Mathematics beginning with Euclid and finishing with Baica. I have the greatest respect for the inspiring work of these great mathematicians before me, who historically paved the way for me to finish this final step and give mathematics this very powerful tool, which is the Generalized Euclidean Algorithm, the Euler System (ES) of the Algebraic Number Theory [23].

The role of these great mathematicians who led the author to the proof of (EFLT) in [13] should not be undermined [45]. All of these great amthematicians aimed at producing the Generalized Euclidean Algorithm (GEA) and to prove its periodicity sometime in their life, and with their seminal work which the author put together, their dreams become a reality and now we have (BGEA) to be the Generalized Euclidean Algorithm.

The application of (BGEA) does not stop here. In many other published papers the author has extended the application of the restricted periodicity of (BGEA) to produce solutions of olmoust all complicated unsolved problems in Algebraic Number Theory.

In conclusion, (BGEA) is very powerful when it becomes periodic. It is as fruitful in n-dimensions as the (EA) is in quadratics [13].

The (BGEA) will dominate mathematics for higher dimension fields in the years to come, exactly as (EA) domunated mathematics for quadratics fields for so many years in the past. In future years it will give a completely new look to the Algebraic Number Theory, whose father was F. Gauss.

In the Modern Era the role of modeling theory in solving application problems has increased tredimendously. The Theory of Algorithms already palys a major role in presentday computer science.

Starting in 1984 (BGEA) opened a new horizont in solveng problems in applied mathematics. The (BGEA) has been used to develop models and then to develop algorithms for their solution.

If the algorithm is fimite, then the problem is solved completely in that model; if it is periodic, the solution does not exist for the tail but is completely solved in the loop; and if the algorithm is not periodic then we use it until we are satisfied with the error.

The future of Applied Mathematics will be to invent mathematical models with their corresponding Computing Algebra (or Arithmetic; i.e. Number Theory) and algorithms (which are the Euler Systems (ES) in that Algebra or Number Theory) in order to solve the necessary Application Problems.

These new models and new algorithms in the Computing Algebra will aid to the transition from abstract to Applied Mathematics.

Important Conclusion Remark

The (ELFLT) proof was overwhelmingly embraced by the American Mathematical Society as being the same as (EFLT), and now we put this group to the test and ask them to use the Elliptic curves to show that, in Elliptic

$$x^2 + y^2 = z^2$$

has the same parametric solutions like in Euclidean

$$x = u^2 - v^2, \quad y = 2\,u\,v, \quad z = u^2 + v^2;$$

with

$$g\,c\,d\,(u, v) = 1$$

for primitive solutions.