13. POSSIBILITIES TO REPRESENT SOME SYMMETRIC FUNCTIONS BY UNIFIED EQUATIONS AND SOME OF THEIR APPLICATIONS IN THE PARATRIGONOMETRY

13.1. Introduction

Many times in practice, especially in Technology, we meet situations when some graphical representations as contours of mechanical parts etc. consist in putting together symmetrically of some curves segments which can't be mathematically modeled by a single equation but by two distinct equations. These curves together are not represented by monotonic functions [18, 19] but they are "fractured" at their contact point (Chapter 3) and in the Classical Mathematics the attached functions to these curves are represented by distinct equations. Examples of this kind are the tooted wheel profiles and the profiles of some cams, etc.

In what follows, we find a way to represent these two equations which commonly are represented by two algebraic symmetric functions, by a unified equation. For example, we will analyze some of such cases, starting with the simplest one, namely the case of a fractured line segment represented by the equation $y = a \cdot x$ (a const.) in the domain $0 \leq x < \lambda$, accepted as the "basic function" (the segment *OM* of Figure 13.1). The "Symmetric Function" of the respective basic function is characteristic to the segment *MN* (corresponding to the domain $\lambda \leq x \leq 2\lambda$) of Figure 13.1.

With this goal in our mind, we will analyze this first case as well as some others in which intervene equations of superior degree and also some paratrigonometric functions.

13.2. The case of the basic function $y = a \cdot x$.

From the beginning we mention that in general the symmetric functions have for symmetric axis the ordonate Oy axis [18]. In order to simplify what follows in our analysis we will accept as the symmetry axis a line parallel with Oy axis situated at a distance λ from it.

In Figure 13.1 is represented the basic function (denoted by y_b), characteristic to the segment *OM* (corresponding to the interval $0 \leq x < \lambda$) and the symmetric function (denoted by y_s), characteristic to the line segment *MN* (corresponding to the interval $\lambda \le x \le 2\lambda$), as we have shown above.

Thus we have:

$$y_b = a \cdot x_b . \qquad (13.1)$$

If we consider that it for $x_b = \lambda$ we have $y_b = m$ the equation (13.1) becomes:

$$y_b = (m / \lambda) \cdot x_b . \qquad (13.2)$$

We mention that by x_b we denoted the current abscises for the functions represented by the equations (13.1) and (13.2) respectively and thus is referred to the segment *OM*.

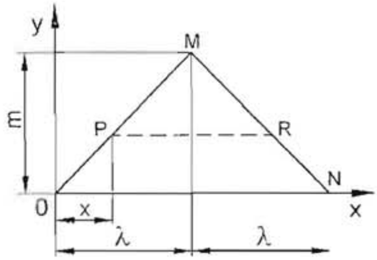

Fig. 13.1. Graphical representation of some line segments *OM* and *MN* symmetrically situated.

As we have shown, the line segment *MN* is symmetric with *OM*, with respect to the vertical line which passes trough *M*. The general form of the line equation where the segment *MN* belongs is:

$$y_s = b \cdot x_s + c . \qquad (13.3)$$

We mention that by x_s we denoted the current variable referred to the segment *MN*. The constants *b* and *c* can be determined using the values for y_s and x_s at the limits of the interval $\lambda \le x \le 2\lambda$; thus for $x_s = \lambda$ we have $y_s = m$, and for $x_s = 2\lambda$ we have $y_s = 0$. Introducing the above values in the equation (13.3), we obtain $b = -(m / \lambda)$ and $c = 2m$ and the respective equation becomes:

$$y_s = 2m - (m / \lambda) \cdot x_s . \qquad (13.4)$$

The equations (13.2) and (13.4) for the basic function and the symmetric function respectively are different at the first glance, they can't be written under an unified form. In order to do that we see that in Figure 13.1 the value of y_b at the point *P* is equal with the value of y_s at the point *R*. Thus we have:

$$y_{b(P)} = y_{s(R)} . \qquad (13.5)$$

In the other side, the abscise values of the two cases are different, but between them there exist the relation

$$x_{s(R)} = 2 \cdot \lambda - x_{b(P)} . \tag{13.6}$$

This means that these two equations (13.2) and (13.4) could be unified under a general form:

$$y = -(m / \lambda) \cdot x + 2 \cdot m . \tag{13.7}$$

If we can find a way that for the interval $0 \leq x < \lambda$ the sign of the factor (m / λ) could change from – (minus) to + (plus) and to dismiss the term $2m$.

This think is possible if we introduce the factors A and B which we name "Binary Operators" motive which we will further explain. These operators are given by the following relations:

$$A = (x - \lambda) / |x - \lambda| \tag{13.8}$$

$$B = \left[(x - \lambda) + |x - \lambda| \right] / 2|x - \lambda| . \tag{13.9}$$

In the above relations $(x - \lambda)$ represents the algebraic value of the difference between x and λ, and $|x - \lambda|$ represents the absolute value of this difference. We can see that for $0 \leq x < \lambda$, we have $A = -1$, and for $\lambda \leq x < 2\lambda$ we have $A = +1$. Now, what happen when $x = \lambda$? In some Mathematical Works [18] is shown that a function of x, as it is A, does not have a limit for $x = \lambda$. In other words saying, for $(x - \lambda - \varepsilon)$, where ε is a positive value infinitely small, we have $A = -1$ and for $(x - \lambda + \varepsilon)$ we have $A = +1$. On the other side we observe that if in the relation (13.8) we make $x = \lambda$, for A appears an indetermination $A = 0/0$. If we apply l'Hôspital Rule to eliminate the indetermination for the point M, we will have $d(x - \lambda) / d|x - \lambda| = 1$ and thus for this point the operator $A = +1$. With another way saying the point M belongs to the segment MN, and the segment OM approaches to the point M up to its immediate neighborhood. Thus we can say that $A = -1$ for $0 \leq x < \lambda$ and $A = +1$ for $\lambda \leq x < 2\lambda$. For this reason and because of what we have mentioned before, we used these corresponding limits.

Applying the same reasoning from above for the operator B case it results that for $0 \leq x < \lambda$ we have $B = 0$, and for $\lambda \leq x < 2\lambda$ we have $B = +1$.

We named A and B as "Binary Operators" since for these two value intervals of x above mentioned, they change their value from -1 to $+1$ (for operator A case) and from 0 to +1 (for operator B case) or inversely (see Table 13.1 in continuation).

These facts being established in order to give to the formula (13.7) a unique general form for the two analyzed symmetric functions, we use in a proper way these two operators A and B and we obtain the equation:

$$y = -A \cdot (m / \lambda) \cdot x + 2 \cdot B \cdot m . \tag{13.10}$$

For the interval $0 \le x < \lambda$, we have $A = -1$ and $B = 0$ and the equation (13.10) becomes the equation (13.2). For the interval $\lambda \le x < 2\lambda$ we have $A = 1$ and $B = 1$ and the equation (13.10) becomes the equation (13.7).

These operators mentioned above (A or B) were used by the authors in the Chapter 9.

13.3. The case of the circle segments (quarters)

In Figure 13.2 we represent as the basic function graph the quarter of the circle OM. The graph of the symmetric function is the quarter of the circle M.

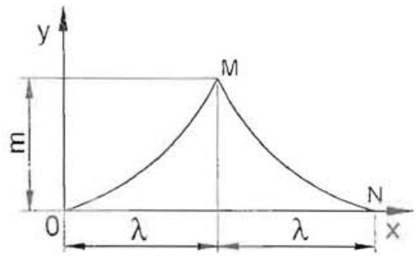

Fig. 13.2. Circle quadrants *OM* and *MN* symmetrically situated.

The equations of the corresponding functions are:

$$y_b = \lambda - \left(\lambda^2 - x^2\right)^{1/2} \tag{13.11}$$

$$y_s = \lambda - \left[\lambda^2 - (2\lambda - x)^2\right]^{1/2}. \tag{13.12}$$

Proceeding as in the previous case, discussed in Subchapter 13.2, we obtain this unique equation

$$y = \lambda - \left[\lambda^2 - x^2 - 4B\lambda(\lambda - x)\right]^{1/2}. \tag{13.13}$$

For $0 \le x < \lambda$ we have $B = 0$ and the equation (13.13) becomes the equation (13.11) and for $\lambda \le x < 2\lambda$ we have $B = 1$ and the equation (13.13) becomes the equation (13.12).

13.4. The hyperbolas case

We accept hyperbola as basic function represented by the equation:

$$y_b = \left(a \cdot x^2 + b \cdot x\right)^{1/2}. \tag{13.14}$$

This hyperbola passes through the origin and does not have in equation (13.14) any constant unassociated with the variable x.

Beside the point with the coordinates $(x=0;\ y_b=0\cdot c)$, we consider that the hyperbola also passes through the point of the coordinates $(x=0.5\cdot\lambda;\ y_b=0.6\cdot m)$ – see Figure 13.3, curve *OM*.

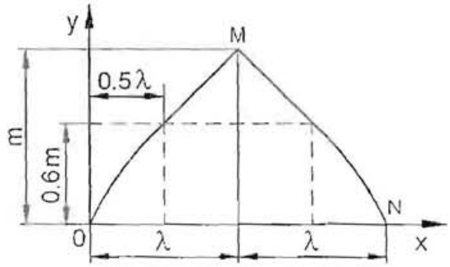

Fig. 13.3. Hyperbola segments *OM* and *MN* symmetrically situated.

In these conditions the equation (13.14) becomes:

$$y_b=\left[0.56\cdot(m/\lambda)^2\cdot x^2+0.44\cdot\left(m^2/\lambda\right)\cdot x\right]^{1/2}. \tag{13.15}$$

Proceeding, as in the previous cases, now for the segment *MN* of the hyperbola, symmetric with the segment *OM*, we have the equation

$$y_s=\left[0.56\cdot(m/\lambda)^2\cdot x^2-2.68\cdot\left(m^2/\lambda\right)\cdot x+3.12\cdot m^2\right]^{1/2}. \tag{13.16}$$

In order to make the unification of equations (13.15) and (13.16) possible we write the equation (13.15) under the form:

$$y_b=\left[0.56\cdot(m/\lambda)^2\cdot x^2-2.68\cdot(m^2/\lambda)\cdot x+3.12\cdot(m^2\lambda)\cdot x\right]^{1/2}. \tag{13.17}$$

We see that in the equations (13.16) and (13.17) the first two terms are identical. Thus, we need operators of type *A* and *B* from above to obtain solutions to make active the third term from each of these equations.

In this way we can write the unified equation for the symmetric functions afferent to the hyperbolic segments *OM* and *MN* as:

$$\begin{aligned}y=\Big[&0.56\cdot(m/\lambda)^2\cdot x^2-2.68\cdot(m^2/\lambda)\cdot x+\\&+3.12\cdot(m^2\lambda)\cdot(B-A)\cdot x+3.12\cdot B\cdot m^2\Big]^{1/2}\end{aligned} \tag{13.18}$$

For the interval $0\le x<\lambda$, when $B=0$ and $A=-1$ thus $(B-A)=1$, the equation (13.18) becomes the equation (13.15). For the interval $\lambda\le x<2\lambda$, when $B=1$ and $(B-A)=0$, the equation (13.18) becomes the equation (13.16).

In Table 13.1 we give synthetically all the possibilities in the change of the algebraic signor or the annihilation of the binary operators A and B which can appear alone or together in our present analysis.

Table 13.1

The values of the Binary Operators A and B

The interval	The operator					
	A	$-A$	B	$-B$	$(A-B)$	$(B-A)$
$0 \le x < \lambda$	-1	$+1$	0	0	-1	$+1$
$\lambda \le x \le 2\lambda$	$+1$	-1	$+1$	-1	0	0

13.5. The case of the cubic parabolas

We accept as a basic function a cubic parabola having as equation

$$y_b = a \cdot x^3 + b \tag{13.19}$$

and also we accept that for $x = 0$; $y_b = 0.1 \cdot m$, thus $b = 0.1 \cdot m$. For $x = \lambda$ we will have

$$y_b = a \cdot \lambda^3 + 0.1 \cdot m \tag{13.20}$$

from where $a = 0.9 \cdot m / \lambda^3$ and thus the equation (13.19) becomes

$$y_b = (0.9 \cdot m / \lambda^3) \cdot x^3 + 0.1 \cdot m. \tag{13.21}$$

For the interval $0 \le x < \lambda$ to this equations is corresponds the segment QM of Figure 13.4.

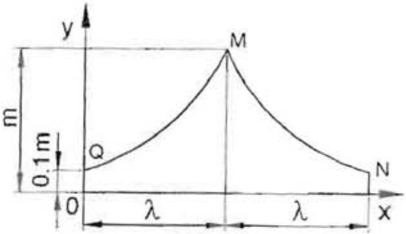

Fig. 13.4. Cubic parabola segments QM and MN symmetrically situated.

The symmetric parabolic segment MN (in the interval $\lambda \le x < 2\lambda$) is represented by the equation

$$y_s = (0.9 \cdot m / \lambda^3) \cdot (2\lambda - x)^3 + 0.1 \cdot m. \tag{13.22}$$

Comparing the equations (13.22) and (13.21) and considering the values of the operators A and B from Table 13.1, it follows that the unified equation of the two symmetric parabolas is

$$y = (0.9 \cdot m / \lambda^3) \cdot (2B\lambda - Ax)^3 + 0.1 \cdot m\,. \tag{13.23}$$

For the interval $0 \le x < \lambda$, we have $A = -1$ and $B = 0$ and thus the equation (13.23) becomes equation (13.21). For the interval $\lambda \le x \le 2\lambda$, we have $A = 1$ and $B = 1$ and thus the equation (13.23) becomes equation (13.22).

13.6. The case of the exponential functions

In this chapter we will analyze two symmetric functions of which the graphical representation is distinct from those of the previous chapters, in the way that in the basic function the value of y decreases at the same time when the value of x increases.

Thus we accept the case of two exponential functions when the basic function has the form:

$$y_b = -a \cdot e^x + b\,. \tag{13.24}$$

Now we impose the following conditions: for $x = 0$ we have $y_b = m$, and for $x = \lambda$ we have $y_b = 0$.

If we introduce these constants a and b in the equation (13.24) we obtain:

$$y_b = [m / (e^\lambda - 1)] \cdot (e^\lambda - e^x)\,. \tag{13.25}$$

This equation is valid for the segment QM of Figure 13.5.

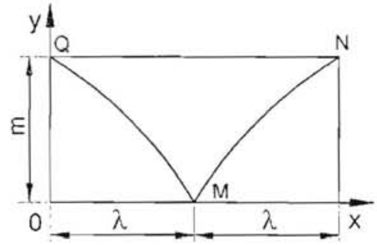

Fig. 13.5. Exponential Functions segments *QM* and *MN* symmetrically situated.

Using a similar reasoning as above, for the segment MN (symmetric with QM) we obtain the equation:

$$y_s = [m / (e^{-\lambda} - 1)] \cdot [e^{(\lambda - x)} - 1]\,. \tag{13.26}$$

Comparing the equations (13.25) and (13.26) and using the operators A and B, we obtain the following unified equation for the two symmetric functions:

$$y = [m/(e^{-A\lambda} - 1)] \cdot [e^{(\lambda - Bx)} - e^{(B-A)x}]. \qquad (13.27)$$

We see that for the interval $0 \le x < \lambda$, the equation (13.27) becomes equation (13.25) and for the interval $\lambda \le x \le 2\lambda$, the equation (13.27) becomes equation (13.26).

13.7. An application in the Paratrigonometry

In chapter 11 we analyzed the case of a representation by a paratrigonometric function (raised to a variable power) of the electrical current i in the rotor of an electrical machine for some conditions of its operation.

In this chapter we give a mathematical model for a portion of this curve with the values of the angle α (rotation angle of the rotor) comprised in the domain $0^\circ \le \alpha < 90^\circ$. The corresponding curve is represented in Figure 13.6 and its portion in the domain mentioned above is comprised between the points O and M, in order to enter in the same notation system used in this paper. In other words, we represent the basic curve i_b by the curve OM.

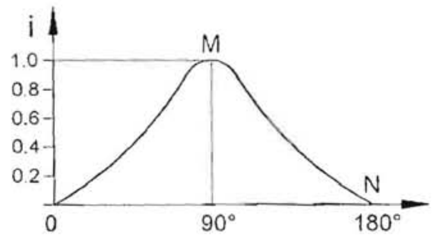

Fig. 13.6. Symmetric Paratrigonometric Function.

For simplification, i was represented in a such a way that its maximum value (for $\alpha = 90^\circ$) is 1 (one).

From Chapter 11 we remember that i and i_b respectively, in our case is given by the relation

$$i_b = (sq\,\alpha)^p \qquad (13.28)$$

where $sq\,\alpha$ is the function "sinus paratrigonometric of order $k = 1$ of the angle α", which is the same thing as "sinus quadratic of the angle α". In its turn, p is a function of the angle α, given by the relation

$$p = 1.3 - 2.245 \cdot 10^{-4} \cdot (\alpha - 10^\circ). \qquad (13.29)$$

Since, as we have shown, the relation i_b is valid for values of the angle α comprised in the domain $0^\circ \le \alpha < 90^\circ$, in the Chapter 11 was mentioned that, for the domain $90^\circ \le \alpha \le 180^\circ$, the form of the curve is obtained applying the trigonometric rules for the supplementary angles.

In what follows, for the domain $90^\circ \le \alpha \le 180^\circ$, we will solve this problem applying the method developed in the previous subchapters.

Thus, for the portion *MN* of the curve in the function *i*, that is for the symmetric portion *OM* we will have the relation

$$i_s = [sq(180^\circ - \alpha)]^\sigma . \tag{13.30}$$

where

$$\sigma = 1.3 - 2.245 \cdot 10^{-4} \cdot (170^\circ - \alpha) . \tag{13.31}$$

Using the binary operators of the type *A* and *B* mentioned above, we can write this unified relation for *i*, which is valid for the entire interval $0^\circ \le \alpha \le 180^\circ$, under this form:

$$i = (B^\circ - A^\circ) \cdot i_b + B^\circ \cdot i_s \tag{13.32}$$

or using the relation (13.28) and (13.30):

$$i = (B^\circ - A^\circ) \cdot (sq\,\alpha)^p + B^\circ \cdot [sq(180 - \alpha)]^\sigma \tag{13.33}$$

where A° and B° are the binary operators *A* and *B*, conform the relations (13.8) and (13.9) from above where the measure x was replaced with α [°], and the measure λ was replaced with 90° compatible with our case.

For the interval $0^\circ \le \alpha < 90^\circ$ we will have $(B^\circ - A^\circ) = +1$ and $B^\circ = 0$ (see Table 13.1), and thus the relation (13.33) becomes the relation (13.28). For the interval $90^\circ \le \alpha \le 180^\circ$ we will have $(B^\circ - A^\circ) = 0$ and $B^\circ = +1$, and thus the relation (13.33) becomes the relation (13.30).

13.8. Conclusions of Chapter 13

From what we have discussed above we can mention some important conclusions:

13.8.1. There exist some situations when some symmetric functions appear and they have graphical representations symmetric with respect to some symmetric axis. These functions were before represented with distinct equations.

13.8.2. Introducing some special mathematical operators named "binary operators" we can relatively easy formulate unified equations for the symmetric functions. The different values of the respective operators (+1, –1 or 0) introduced in the unified equations will particularize them, making them valid for the "Basic function" respectively for the "Symmetric function".

13.8.3. We analyzed for example, functions of degree 1, 2 and 3 as well as exponential functions and also a paratrigonometric function.