The above mentioned trigonometries (IT, QT, TT, C, UT, PT) can be considered "components" of PRT. They are pointed out in Table 6.1, where we represent the entire structure of PRT.

The "categories" as part of PRT, mentioned in Table 6.1 are separated by the Quadratic Trigonometry (QT). This border component between IT and TT represents itself a "category", since there are some symmetries regarding basic trigonometric functions (BTFs) relate to QT, as we will se as follows.

Table 6.1

The Paratrigonometry (PRT) structure

The name of PRT structure	The value domain of the order k	BTF form	The category as part of PRT	The generic denomination
Infratrigonometry (IT)	$0 \le k < 1$	Rhombus with curves sides and the concavity opposite to the reference 0*	Infratrigonometry (IT)	Paratrigonometry (PRT) $0 \le k \le \infty$
Quadratic Trigonometry (QT)	$k = 1$	Rhombus with straight sides (Rhombus with "Paratrigonometric mirrors")	Quadratic Trigonometry (QT)	
Transtrigonometry (TT)**	$1 < k < 2$	Rhombus with curves sides and the concavity towards the reference 0*	Extratrigonometry (ET) $1 < k \le \infty$	
Classical Trigonometry (CT)	$k = 2$	Trigonometric circle		
Ultratrigonometry (UT)	$2 < k \le \infty$	Rhombuses with curved sides and the concavity towards the reference 0*		

* The coordinate system Ox – Oy origine.

** Polygonal Trigonometry (PT) is referred to the domain $1 < k < 2$, but k is variable in function of the sides number in the Trigonometric Polygon.

Having this in consideration, the formulas (6.1) and (6.2) from above receive a specific expression to every component of PRT which we choose to refer to. Thus, for example, in TT they are expressed in the following way:

$$|st_k \alpha|^k + |ct_k \alpha|^k = 1 \tag{6.3}$$

$$tgt\,\alpha = tg\,\alpha \tag{6.4}$$

The index k was attached to the notation of the functions "sine", "cosine" etc., to define its order, and the introduction in formula (6.1) of the absolute values of the respective functions was imposed by the validity need of these formulas for odd and fractional values of k.

Evidently that for IT, in formula (6.1) we will have the notations $si_k\,\alpha$, $ci_k\,\alpha$, for UT the notations $su_k\,\alpha$, $cu_k\,\alpha$ etc.

In the light of what we have said before, in CT formula (6.1) is written under the known form:

$$\sin^2 \alpha + \cos^2 \alpha = 1 \tag{6.5}$$

and in QT this has the form:

$$\left|sq\,\alpha\right| + \left|cq\,\alpha\right| = 1 \tag{6.6}$$

where $sq\,\alpha$ is "quadratic sine of angle α " and $cq\,\alpha$ is "quadratic cosine of angle α ".

The formulas (6.2) and (6.4) respectively have a general character and thus we can write:

$$tpr_k\,\alpha = tgt_k\,\alpha = tgi_k\,\alpha = tgu_k\,\alpha = tq\,\alpha = tp\,\alpha = tg\,\alpha\ . \tag{6.7}$$

From above mentioned chapter regarding the trigonometric formulas analyzed, we see that the principal trigonometric functions of IT ($si_k\,\alpha$, $ci_k\,\alpha$) are characterized by the fact that their graphic representation have an "Arabian Archivolt". This thing is valid and for the principal trigonometric functions in TT case and thus also, for the border between IT and TT, namely for QT. the curves that represent these functions with respect to α are "cusp" points for $\alpha = \pi/2$, $\alpha = 3\pi/2$ etc. in the "sine" functions case, and respectively $\alpha = 0$, $\alpha = \pi$ etc., for "cosine" function.

The principal trigonometric functions in UT and CT inclusively, have $spr_k\alpha$ and respectively $cpr_k\alpha$ curves with a monotonous variations, without cusps. These functions make an exception for $k = \infty$ in UT. In this case these cusps appear for $\alpha = \pi/4$, $\alpha = 3\pi/4$ etc., if we refer to $spr_\infty\alpha$. Evidently, these cusps appear for same α values for the case of function $cpr_\infty\alpha$ also (see Chapter 5).

Therefore, from the point of view of the principal trigonometric functions graphical form, CT (k = 2) represents a border domain. Up to this ($0 \le k < 2$) the curves $spr_\infty\alpha$ and $cpr_\infty\alpha$ have cusps and moreover they are inscribed in the interior of the curves $spr_\infty\alpha = \sin\alpha$ and $cpr_\infty\alpha = \cos\alpha$. Starting with k = 2, thus for $2 \le k < \infty$, the respective curves do not have cusps. For the extreme case $k = \infty$, these cusps appear again. All these curves for the entire domain $2 < k < \infty$ are circumscribed to the characteristics curves for k = 2, thus the classical sinusoid.

Regarding the Basic Trigonometric Figures (BTFs) – Figure 6.1 – we mark the fact that in IT case, for $0 < k < 1$, these have a form of rhombuses with curved sides, and the concavity in the opposite direction to the reference point O of the coordinate axis Ox – Oy; for the limit situation $k = 0$, BTF consist on the cross OM – OP – ON – OQ.

For TT and UT cases ($1 < k \le \infty$), BTFs form is of some rhombuses with curved sides and concavity to the direction of the reference point O, thus in an opposite sense to the concavity of the rhombuses with curved sides in IT. At the limit, when $k = \infty$, the curvature of these sides becomes extreme, they fragments in the points A, B, C and D. In Figure 6.1, in this way, BTF for $k = \infty$ becomes the square ABCD.

In consequence, it is to notice the fact that, from the BTFs point of view, the form of these figures can be part of two categories, namely: first, the one characteristic to IT and the second, characteristic to TT and UT. The limit between these two categories is QT (k = 1) where BTF is the rhombus with straight sides (without concavity) QMNP.

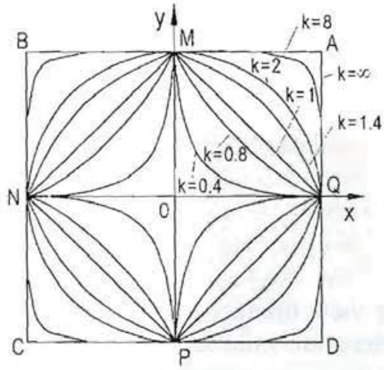

Fig. 6.1. The Basic Trigonometric Figures (BTFs) aspect in the entire Paratrigonometric (PT) domain ($0 \le k \le \infty$).

In the second category, a special case is the characteristic "trigonometric circle" in CT ($k = 2$).

Finally, we can say that, from the point of view of the graphical representation of $spr_k\alpha$ and $cpr_k\alpha$ functions, the border between these two distinct zones is the trigonometric circle, and from the point of view of BTFs, the border between the two distinct zones is the trigonometric rhombus with the straight line in QT. The zone between these two borders characterized by $1 < k < 2$, we named it as Transtrigonometry (TT) – see Chapter 3.

For our analysis which we carry on, because of some reasons which we will explain in detail as follows, we will have in our view as a guide the trigonometric rhombus with straight sides (BTF in QT). This divides the entire domain $0 \le k \le \infty$ in two areas, namely: IT ($0 \le k < 1$) and respectively TT ($1 < k < 2$) together with UT ($2 < k \le \infty$), including CT ($k = 2$). We named these last two trigonometries (TT and UT) together with the "Extratrigonometry" (ET) which, regarding to Figure 6.1, takes possession of the entire zone exterior to the rhombus QMNP ($k = 1$).

Regarding Table 6.1, we have to mention that the reasons for which we name the trigonometric rhombus in CT as the "Paratrigonometric mirror" will be explained here below. So we will analyse some noticed particularities of symmetries regarding BTFs of PRT.

6.3. The Basic Trigonometric Figures (BTFs) of Paratrigonometry and their implicit characteristic equations

In Figure 6.1, we represented the most important BTFs of PRT, namely: the Cross ($k = 0$), the Trigonometric Rhombus with straight sides MNPQ ($k = 1$), the Trigonometric Circle ($k = 2$) and the Trigonometric Square ABCD ($k = \infty$). Also, there are represented some of BTFs of IT ($k = 0.4$ and $k = 0.8$), TT ($k = 1.4$) and UT_n($k = 8$).

From their analysis we observe two symmetries, namely:

a) The curve symmetry in relation with the two coordinate axis for a specific value of k, which is found in one of the trigonometric quadrants;

b) The symmetry of a such curve situated "above" to the rhombus side QM (if we refer to the first quadrant) with a corresponding curve from the space "below" the respective side. In another way saying, BTFs from ET have symmetries in IT and conversely. The line QM is its own symmetry and all other symmetries are related to this one like being a mirror. For this reason we named BTF for QT ($k = 1$) "The Rhombus with paratrigonometric mirrors" (see Table 6.1), having in our view the fact that everything what we have said before regarding to the first quadrant are valid for the other (II – IV) quadrants, also.

In order to analyse from the mathematical point of view these symmetries, we apply to the formula which represent the side in the first quadrant – for simplicity – of BTF.

This formula was represented and explained in Chapter 3 and used in Chapter 4 and 5 and it is valid for the entire domain $0 \le k \le \infty$ and thus it represents a fundamental formula in PRT.

It express the relation between the coordinates x and y of the system Ox – Oy in which BTF is represented and has the form:

$$y = \left(1 - |x|^k\right)^{1/k}. \tag{6.8}$$

In Figure 6.2 there are represented the sides from the first quadrant of BTFs for $k = 1$; $1 < k < \infty$ (of ET domain) and $0 < \kappa < 1$ (of IT domain). We denoted by the Greek symbol κ the order value of the function from IT (to distinct from k of ET) for a reason which we will soon speak about it.

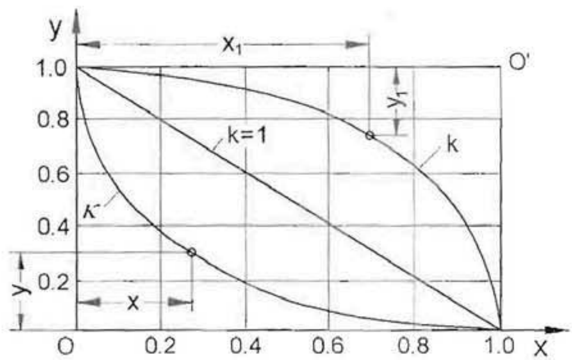

Fig. 6.2. Symmetric BTFs given the mirror $k = 1$, represented in the first quadrant.

In order to prove the symmetry of "a)" above it is sufficient to set formula (6.8) under the form:

$$x = \left(1 - |y|^k\right)^{1/k}. \tag{6.9}$$

The symmetry of formulas (6.8) and (6.9) direct us to conclude that there exists BTFs symmetry in relation with Ox and Oy coordinate axis.

Since we only refer to the first quadrant, as we have mentioned before, in formulas (6.8) and (6.9) it is no more in need to use "absolute values" for x and y (always positive in this quadrant) and the corresponding formulas can be written under a simpler form:

$$y=\left(1-x^k\right)^{1/k} \tag{6.10}$$

$$x=\left(1-y^k\right)^{1/k}. \tag{6.11}$$

In what follows, we will use formula (6.11) applied in ET, as we have mentioned before. On the other side, for IT this formula will have the form:

$$y=\left(1-x^\kappa\right)^{1/\kappa}. \tag{6.12}$$

To analyze the symmetry type mentioned at the point "b)" above, we consider that the curves characterized by k and κ of Figure 6.2 are symmetric with respect to the line $k = 1$. In another way said, the curve κ is the "mirror" image (with respect to $k = 1$) of curve k, and conversely.

We denote the present coordinate of curve κ with x and y, and the present values of curve k with x_1 and y_1. We apply formulas (6.10) and (6.12) for the curves k and κ case. For curve k case we have:

$$y_1=\left(1-x_1{}^k\right)^{1/k} \tag{6.13}$$

and for curve κ case, formula (6.12) is also valid.

Referring to Figure 6.2, we can write $x_1=(1-x)$ and $y_1=(1-y)$. Introducing x_1 and y_1 in formula (6.13) in this way expressed as functions of x and y and using formula (6.12) to express y as function of x, we obtain the relation:

$$1-\left[1-(1-x)^k\right]^{1/k}=\left(1-x_1{}^\kappa\right)^{1/\kappa}. \tag{6.14}$$

If we take logarithm of relation (6.14) we obtain the following banding formula between κ and k:

$$\kappa=\left[\ln\left(1-x^\kappa\right)\right]/A \tag{6.15}$$

where

$$A=\ln\left\{1-\left[1-(1-x)^k\right]^{1/k}\right\}. \tag{6.16}$$

We see that from formula (6.15) we can not find κ explicitly. In other words, formula (6.15) is implicitly given regarding κ (and also k).

In any case, the values of κ and k characterize two symmetric curves in ET and respectively in IT given the "mirror" having $k = 1$. If we symbolize the symmetry "status" by "Σim", we can write $\Sigma\text{im }k \to \kappa$. Evidently, we can also write $\Sigma\text{im }\kappa \to k$, or simply $\Sigma\text{im }k \leftrightarrow \Sigma\text{im }\kappa$.

Since κ and k respectively can be find in an implicit form in the formula (6.15) to determine κ as a function of k we proceed graphically as we will continue to show.

On the left side of the equality sign in formula (6.15) we replace κ by z. In this way we will obtain the following formula which express z as a function of κ:

$$z = \left[\ln\left(1 - x^{\kappa}\right)\right] / A \tag{6.17}$$

where A is given by formula (6.16).

In this way $z = z(\kappa)$ have both parameters k and κ. Giving various values for k (in the domain $1 < k < \infty$) and for κ (in the domain $0 < \kappa < 1$) we obtain various curves representing $z = z(\kappa)$, as it can be seen in the Figure 6.3.

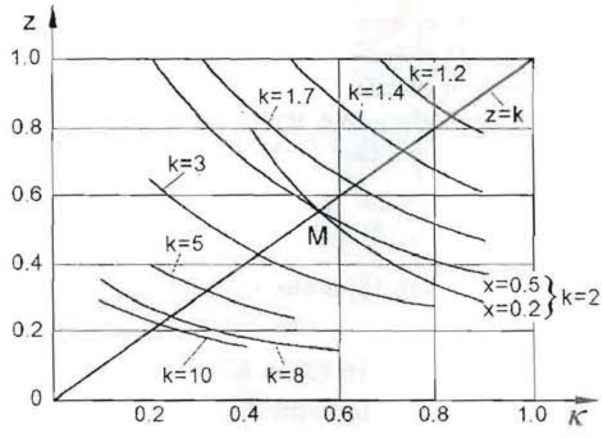

Fig. 6.3. The representation of the function $z(\kappa)$ for $x = 0.5$ (in the case $k = 2$), also for $x = 0.2$ having as parameter distinct values for k.

If for a certain value of k we live two values for x, the curves of the function $z(\kappa)$ intersect in a point whose coordinates represent the solution of the problem, namely $z = \kappa$. In Figure 6.3, M is a such point which represent the intersection of the curves $z(\kappa)$ for $k = 2$ and for the values of x = 0.5 and x = 0.2 respectively. The result is $\kappa = 0.56$. Thus we can write:

$$\Sigma\text{im }(k = 2) \leftrightarrow \Sigma\text{im }(\kappa = 0.56). \tag{6.18}$$

Any other curve of the function $z(\kappa)$ for $k = 2$ corresponding other values of x will pass through the point M. This is because for any value of x the curve for κ of Figure 6.2 is symmetric with respect to the "mirror" $k = 1$, of the curve characterized by k. Evidently, the converse is valid also, as it is shown by formula (6.18).

In Figure 6.3 we see that the point M is on the line OM which represents the bisector of the right angle formed by the coordinate axis Ox – Oy. In fact the equation of this line is even $z = z(\kappa)$. The intersection of this line with other curves, for various values of k (see Figure 6.3), will give the solutions for Σim $k \to \kappa$.

6.4. The semiempirical equation for the basic symmetric figures from Paratrigonometry

With values for $\kappa = \Sigma\text{im } k$, from Figure 6.3, we construct the curve $\kappa = \varphi(k)$ represented in Figure 6.4.

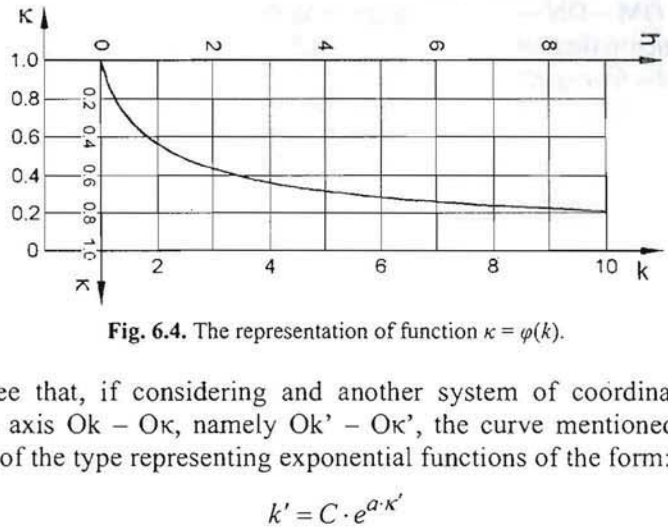

Fig. 6.4. The representation of function $\kappa = \varphi(k)$.

We see that, if considering and another system of coordinates than the system with axis Ok – Oκ, namely Ok' – Oκ', the curve mentioned before will have a form of the type representing exponential functions of the form:

$$k' = C \cdot e^{a \cdot \kappa'} \tag{6.19}$$

In order to have this function (6.19) expressed in the κ and k coordinates we replace κ' and k' by $\kappa' = (1-\kappa)$ and $k' = (1-k)$.

After making these substitutions, we apply again logarithm to the relation (6.19) and going the necessary operations for simplification we arrive to the following formula:

$$\kappa = \varepsilon - \left[\ln(k-1)\right]/a. \tag{6.20}$$

By trying some values for constants ε and a, and comparing the resulting values from formula (6.20) with those of Figure 6.4, we obtain $\varepsilon = 0.56$ and $a = 6$. With these values, formula (6.20) becomes:

$$\kappa = 0.56 - \left[\ln(k-1)\right]/6. \tag{6.21}$$

This formula (6.21) represents, with a high precision degree, this function $\kappa = \varphi(k)$ for the values of *k* in the domain $1.075 < k < 10$.

Formula (6.21) is not applicable for *k* = 1, but returning to formula (6.17) we conclude that for *k* = 1 introduced in formula (6.15) we have $z = \kappa$ only if $\kappa = 1$. That is

$$\Sigma\text{im}\,(k=1) \leftrightarrow \Sigma\text{im}\,(\kappa=1). \tag{6.22}$$

This last relation represents the mathematical expression of the "mirror" in the Paratrigonometry.

For the extreme cases marked by values *k* = 0 and respectively *k* = ∞ we apply the results of Chapters 4 and 5 mentioned before. Thus, in Chapter 4 we proved that, in IT, for *k* = 0 (now denoted $\kappa = 0$), the BTF is the Cross (OQ – OM – ON – OP, Figure 6.1), and in Chapter 5 we proved that, in UT, for *k* = ∞, the BTF is the square ABCD of Figure 6.1.

From the geometrical point of view it can be seen in Figure 6.1 that the Cross OQ – OM – ON – OP is symmetric to the square ABCD, in relation with the paratrigonometric rhombus of the mirrors. The same thing we can see in Figure 6.2, referring to the first quadrant. Using the above symbols, this thing can be expressed as:

$$\Sigma\text{im}\,(k=\infty) \leftrightarrow \Sigma\text{im}\,(\kappa=0). \tag{6.23}$$

Again from the analysis of formula (6.21) we can see that for *k* = 2 we obtain $\kappa = 0.56$ as we shown in the previous chapter and we have found it using relation (6.18).

6.5. Conclusions of Chapter 6

6.5.1. All the "Trigonometries" which we anlysed in the previous Chapters 1, 2, 3, 4 and 5 together with the Quadratic Trigonometry (QT) [1] and the Classical Trigonometry (CT) can be comprised in the notion of the Paratrigonometry (PRT) developed in Subchapter 6.2 here above.

The PRT structure and its relation with all the others trigonometries is given in Table 6.1. The basic relations from PRT, (6.1) and (6.2) can be applied in the case of all mentioned trigonometries, distinguishing themselves by the values for the "order" *k*.

This is also mentioned in Table 6.1, where we point out other classification elements, namely the Basic Trigonometric Figures (BTFs), as could be the Trigonometric circle in CT, Trigonometric rhombus with straight sides in QT etc.

6.5.2. Regarding BTFs, in PRT we established some BTFs symmetries, such these among BTFs from the Extratrigonometry (ET) with $1 < k \le \infty$ and BTFs from the Infratrigonometry (IT) with $0 \le k < 1$ in relation with the "Paratrigonometric mirror" of QT with *k* = 1 amply analyzed in Subchapter 6.3 and 6.4 here above.

Thus, in Subchapter 6.3 we established a semiempiric explicite equation for the respective symmetries. For a better expression of these symmetries, in IT case the order of the trigonometric functions was denoted by κ and the k notation was reserved for ET. We introduced the symbol "Σim", and the symmetry between a BTF from ET (of order k) and a corresponding BTF of than from ET of order κ from IT, was denoted by Σim $k \leftrightarrow$ Σim κ, the sign $\leftrightarrow$ indicating the respective relation reciprocally.