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MORE UNITS FROM THE PERIODICITY OF AN ALGORITHM

by Malvina Baica

ABSTRACT. The author uses the periodicity of a modified Jacobi-Perron Algorithm, called ACF, in order to derive some more units in algebraic number fields. In one of his papers Neubrand considered parametric fields like,

$\{ Q(w),\ H(N,w) - bN^{l} = 0, N \in \mathbb{Z}$ where $w = \sqrt[n]{aN^{n} + bN^{l}}$, $a=\alpha^{n}$, $\sigma \in \mathbb{Z}$, $b \mid \alpha^{n}$;

$$H(N,w) = \prod_{i=0}^{n-1} h_i(N,w)^{s_i} \quad \text{with} \quad h_i(N,w) = w^{s_i} + \alpha_{i1} N w^{s_i - 1} + \dots + \alpha_{is_i} N^{s_i},$$

and without using an algorithm derived units of the form:

$$\varepsilon_{jk} = \frac{h_j(N,w)^{s_k}}{h_k(N,w)^{s_j}}$$

Under certain conditions the units from ACF become some (not all) of the above units. In this way, ACF satisfies partially, Hilbert's demand for the disclosure of units.

0. INTRODUCTION. In this paper the author obtains some more units for certain algebraic number fields from the periodicity of ACF algorithm. In a previous paper, the author developed this new algorithm ACF, proved its periodicity for some algebraic number fields and derived some known units in those corresponding fields with a unified method. The units in this paper were derived by Neubrand in his dissertation without using an algorithm.

As is known, the problem of finding the multiplicative group of units in any algebraic field F over the field Q of rationals, is still a difficult open question. Since Dirichlet proved that the group G of units of a number field is finitely generated, many mathematicians have invented ingenious algorithms to calculate fundamental system of units (the elements of the basis are called fundamental units). With all these efforts, Hilbert's demand to construct a universal algorithm by means of which the expansion of any real algebraic number becomes periodic, thus enabling a complete system of fundamental units in the corresponding field to be calculated, still is and probably will remain for centuries an open difficult question.

A breakthrough in finding units from a periodic algorithm took place when Bernstein and Hasse [7,8] and later Bernstein [4] considered algebraic number fields of the form:

A) $Q(w)$, $w^n = D^n+d$, $D,d\in N$, $d|D$ and $D \geq d(n-2)$

B) $Q(w)$, $w^n = D^n-d$, $D,d\in N$, $d|D$ and $D \geq 2d(n-1)$

and proved that some units in the corresponding fields are given by

$$e_s = (w-D)^{-s}(w^s-D^s), \text{ where } s>1 \text{ and } s|n.$$

It was conjectured by Bernstein and Hasse and it was proved by Stender [14] that in the case A, for $n=3,4,6$ the units form a maximal independent system of fundamental units under additional assumptions. Whether a unit is fundamental or not requires investigation in each case. Later, Halter-Koch and Stender [10] and Halter-Koch [9] and Neubrand [11], find units for much wider classes of algebraic number fields without an algorithm. Bernstein [5,6] using two different modifications of Jacobi-Perron Algorithms (JPA) derived Halter-Koch and Stender and also Halter-Koch units from their periodicity.

In [1] the author developed a periodic algorithm over the complex number field (ACF) and derives those many results in the theory of units including Neubrand units by the means of a unified algorithm.

The purpose of this paper is to obtain Neubrand units from the periodicity of ACF.

1. ACF - PRELIMINARIES. We introduce

(1.1)

$$
(1.1)\quad \begin{cases} P(x) = \left(\prod_{i=1}^{k} (x^{s_i} - D_i^{s_i}) - d\right) \\ k \geq 2,\ s_i \geq 1;\ D_i - \mathbb{N};\ d \mid D_i\ ; \\ d \in \mathbb{Z};\ i = 1,2,\ldots,k;\ |d| \geq 1\ ; \\ 0 < D_1 < D_2 < \ldots < D_k. \end{cases}
$$

$P(x)$ is irreducible over the field of rationals [1] and has at least one real root $\bar{w}$ of the form

$$
(1.2)\qquad w = \sqrt[n]{D^n + d}\ ,\ d \mid D,\ D \in \mathbb{N},\ d \in \mathbb{Z},\ n \geq 2\ ;
$$

Denoting

$$
(1.3)\quad \begin{cases} s_1 + s_2 + \ldots + s_k = n, \\ s_i \text{ from } (1.1),\ i = 1, \ldots, k. \end{cases}
$$

We now factorize $P(x)$ and introduce the notation:

$$
(1.4)\qquad P(x) = \prod_{j=1}^{k} (x - D_j)\ (x - \rho_j D_j)\ \ldots\ (x - \rho_j^{s_j - 1} D_j).
$$

$$
(1.5)\quad \begin{cases} P(\bar{w}) = 0;\ \rho_j = e^{\frac{2\pi i}{s_j}}\ ; \\ \left\{\ldots, D_j, \rho_j D_j, \rho_j^2 D_j, \ldots, \rho^{s_j - 1} D_j, \ldots\right\} \\ = \left\{\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_n\right\};\ j = 1, \ldots, k. \end{cases}
$$

We shall construct an ACF involving the numbers $\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_n$ once a pairing of these numbers with those of the first set of (1.5) has been fixed, we have to cling to this choice during the process of the ACF.

Let us construct the starting vector

$$(1.6)\quad \begin{cases} \bar{a}^{(0)} = \left(f_{1,\,n-1}(\bar{w}),\ f_{1,n-2}(\bar{w}),\ldots,\ f_{1,2}(\bar{w}), f_{2,2}(\bar{w})\right) \\ f_{i,k}(\bar{w}) = \prod_{s=i}^{k}(\bar{w}-\bar{D}_s)\ , \\ f_{i,i}(\bar{w}) = \bar{w}-\bar{D}_i,\ 1\le i\le k\le n \\ P(\bar{w}) = 0,\ P(x) \text{ from } (1.1). \end{cases}$$

For the generation of the companion vectors we use the formula

$$(1.7)\quad \bar{b}_i^{\ (v)} = \bar{a}_i^{\ (v)}\ (D_1);\ i=1,\ldots,n-1,\ v=0,1,\ldots\ .$$

<u>THEOREM 1</u>. The ACF of $\bar{a}^{(0)}$ from (1.6) with the generating formula (1.7) for the companion vectors is purely periodic and the length of the primitive period of the ACF equals $m=n(n-1)$ for $d \neq 1$ and $m=n-1$ for $d=1$.

<u>COROLLARY 1</u> to Theorem 1. The product of the n-1-st components of the n(n-1) vectors of the primitive period of ACF of $\bar{a}^{\ (0)}$ equals

$$(1.8)\quad d^{-(n-1)}\ \left((\bar{w}-\bar{D}_2)\ (\bar{w}-\bar{D}_3)\ \ldots\ (\bar{w}-\bar{D}_n)\right)^n$$

<u>COROLLARY 2</u> to Theorem 1. The components of the n(n-1) companion vectors of the ACF of $\bar{a}^{\ (0)}$ equal

$$(1.9)\quad \begin{cases} \bar{b}_i^{(v)} = D,\ i=1,\ldots,n-2;\ v=0,1,\ldots,n(n-1) \\ \bar{b}_{n-1}^{(v)} = \bar{D}_1-\bar{D}_i,\ i=2,\ldots,\text{ or} \\ \bar{b}_{n-1}^{(v)} = d^{-1}(\bar{D}_1-\bar{D}_i),\ i=2,\ldots,n. \end{cases}$$

Thus all these companion vectors (1.9) are algebraic integers.

We now turn to gaining units in the field $Q\ (\bar{w},\bar{\rho})$ of type k.

<u>THEOREM 2</u>. A unit in the field $Q\ (\bar{w},\bar{\rho})$ is given by the expression:

(1.10)

Since $(\bar{w}-\bar{D}_1)$

(1.11)

Dividing (1.

(1.12)

Since $\bar{D}_1$ cou

(1.13)

Choosing for

(1.14)

Thus the k un

degree n.

We choos

(1.15)

(1.10) $$\bar{e} = d^{-(n-1)} \left((\bar{w}-\bar{D}_2)\ (\bar{w}-\bar{D}_3)\dots(\bar{w}-\bar{D}_n)\right)^n$$

Since $(\bar{w}-\bar{D}_1)\ (\bar{w}-\bar{D}_2)\dots(\bar{w}-\bar{D}_n) = d$,

(1.11) $$\frac{\left((\bar{w}-\bar{D}_1)\ (\bar{w}-\bar{D}_2)\dots(\bar{w}-\bar{D}_n)\right)^n}{d^n} = 1.$$

Dividing (1.11) by (1.8) we obtain

(1.12) $$\bar{e}_1 = \frac{(\bar{w}-\bar{D}_1)^n}{d} \text{ is a unit in } Q(\bar{w},\bar{\rho}).$$

Since $\bar{D}_1$ could be any number out of the n numbers $\bar{D}_1,\bar{D}_2,\dots,\bar{D}_n$, we have

(1.13) $$\begin{cases} \bar{e}_t = \dfrac{(\bar{w}-\bar{D}_t)^n}{d}, \quad t = 1,\dots,n, \\ \text{are units in } Q(\bar{w},\bar{\rho}) \end{cases}$$

Choosing for $\bar{D}_t = D_1, D_2, \dots, D_k$ from $P(x)$ we obtain

(1.14) $$\begin{cases} \bar{e}_i = \dfrac{(\bar{w}-\bar{D}_i)^n}{d}, \quad i = 1,2,\dots,k, \\ \text{are units in } Q(\bar{w}). \end{cases}$$

Thus the k units (1.14) are units in the real algebraic number fields $Q(\bar{w})$ of degree n.

We choose the e_i units in $Q(\bar{w},\rho_i)$

(1.15) $$\begin{cases} \bar{e}_{i,0} = \dfrac{(\bar{w}-D_i)^n}{d}, \\ \bar{e}_{i,1} = \dfrac{(\bar{w}-\rho_i D_i)^n}{d}, \end{cases}$$

$$\begin{cases} e_{i,2} = \dfrac{(w-p_i^2 D_i)^n}{d}, \ldots, \\ e_{i,s_i-1} = \dfrac{(w-p_i^{s_i-1} D_i)^n}{d}. \end{cases}$$

and if we multiply all the units of (1.15) by each other, we obtain the unit

$$\bar{e}_i = \frac{(\bar{w}^{s_i} - \bar{D}_i^{s_i})^n}{d^{s_i}} \quad (i = 1,\ldots,k) \tag{1.16}$$

are k units in $Q(\bar{w})$.

We must stress that only the units (1.10) were obtained by an ACF.

Apart from slight changes in notation, the proofs and the results here follow the proofs and the results of [1], verbatim.

2. THE STATEMENT OF THE PROBLEM. In his paper which contains some of the results of his dissertation Neubrand [11] studied functional or parametric fields of the form

$$\left\{ Q(w),\ H(N,w) - bN^{\ell} = 0,\ N \subset \mathbb{Z} \right\} \tag{2.1}$$

where

$$w = \sqrt[n]{aN^n + bN^{\lambda}},\ a = \alpha^n, \alpha \in \mathbb{Z},\ b \mid \alpha^n. \tag{2.2}$$

and

$$H(N,w) = \prod_{i=0}^{n-\lambda} h_i(N,w)^{g_i} \tag{2.3}$$

with

$$h_i(N,w) = w^{s_i} + \alpha_{i1} N w^{s_i-1} + \ldots + \alpha_{is_i} N^{s_i} \tag{2.4}$$

Neubrand proved in [11]

<u>THEOREM 3.</u> In functional fields of the form (2.1) with

$$b^{\lambda} \mid \alpha_{ji}^n \text{ for } j=0,1,\ldots,t-1,\ i=1,\ldots,s_j \tag{2.5}$$

the elements

are units with

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where $h_j(N,w)^{s_j}$

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(2.7)

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3. SOLUTION OF

the elements

$$\varepsilon_{jk} = \frac{h_j(N,w)^{s_k}}{h_k(N,w)^{s_j}} \text{ for } j,k = 0,1,\ldots,t-1$$

are units with the

$$(2.6) \qquad \text{Norm} \quad \varepsilon_{jk} = 1.$$

where $h_j(N,w)^{s_k}$, $h_k(N,w)^{s_j}$ are defined as in (2.4).

$$\text{Since} \quad \varepsilon_{jk}^{n} = \frac{h_j^{ns_k}}{(bN^{\ell})^{s_j s_k}} \cdot \frac{(bN^{\ell})^{s_k s_j}}{h_k^{ns_j}} =$$

$$\left(\frac{h_j^n}{(bN^{\ell})^{s_j}}\right)^{s_k} \cdot \left(\frac{(bN^{\ell})^{s_k}}{h_k^n}\right)^{s_j}$$

Then

$$(2.7) \qquad E_j = \frac{h_j(N,w)^n}{(bN^{\ell})^{s_j}}, \quad j = 0,1,\ldots,t-1$$

is also a unit in $Q(w)$, w as in (2.2).

Neubrand's method to derive units is not an algorithmic method. This method is quite different from the other methods and it is algebraic geometric or function theoretic oriented. Using his method later Neubrand [12] got units in quadratic fields.

Nothing is known about the fundamentability of these units and requires further investigations in each case. Stender, who worked in this problem, informed Neubrand that his units ε^* and ε^{**} [12] are also fundamental units.

3. SOLUTION OF THE PROBLEM. In (2.2) let

(3.1) $\qquad bN^{\ell} = d,\ (\alpha N)^{n} = D^{n}$ and $g_i = 1.$

Then since

$$(3.2) \qquad H(N,w) = \prod_{i=0}^{n-1} h_i(N,w)^{g_i} = \prod_{i=0}^{n-1} (w^{s_i} - D_i^{s_i}).$$

(2.1) becomes

$$(3.3) \qquad \begin{cases} Q(w),\, w = \sqrt[n]{D^n + d},\ P(w) = 0, \\ P(x) \text{ as in } (1.1) \end{cases}$$

From here on, it is obvious that under the conditions of (3.1) Neubrand's units can be derived from the periodicity of ACF.

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