15

ISSN [illegible]

NOTES ON NUMBER THEORY AND DISCRETE MATHEMATICS

VOLUME 1 1995 NUMBER 3

Prof. Marin Drinov Academic Publishing House
of the Bulgarian Academy of Sciences
Bl. 6, Acad. George Bonchev Str., Sofia-1113, BULGARIA

NNTDM 1 (1995) 3, 120-134

BAICA'S GENERAL EUCLIDEAN ALGORITHM (BGEA) AND THE SOLUTION OF FERMAT'S LAST THEOREM

MALVINA BAICA
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
THE UNIVERSITY OF WISCONSIN
WHITEWATER, WISCONSIN 53190

ABSTRACT The main intent in this paper is to solve Fermat's last theorem (FLT) using the author's modification of the Jacobi-Perron Algorithm (JPA) which holds for complex fields of any degree n (ACF) (defined later as Baica's Generalized Euclidean Algorithm (BGEA)).

KEY WORDS AND PHRASES
EUCLIDEAN ALGORITHM (abbr. EA)
JACOBI-PERRON ALGORITHM (abbr. JPA)
BAICA'S ALGORITHM IN A COMPLEX FIELD (Abbr. ACF)
(ACF) also named BAICA'S GENERALIZED EUCLIDEAN ALGORITHM (abbr. BGE
HASSE-BERNSTEIN MODIFICATION OF JPA (abbr. HBA)

0. INTRODUCTION

We start with the well known and powerful Euclidean Algorithm (EA). Another interpretation of (EA) which leads to the continued fraction is:

Let the starting vector be $a^0 = (a_1^0) \in R^1$ and a transformation function which is the greatest integer function $[a_1^0]$ as a companion vecter $b^0 = [a_1^0] = (b_1^0)$; then the recursive transformation

$$a^{(v+1)} = (a_1^{(v)} - b_1^{(v)})^{-1} = \frac{1}{(a_1^{(v)} - b_1^{(v)})}$$

applied to these vectors become a sequence $\{a^{(v)}\}$, $v=0,1,\ldots$; which is called the continued fraction interpretation of (EA). Because of the periodicity of (EA) many difficult problems which were still open in fields of higher degree can be solved in quadratic fields. For example, by using this algorithm it is easy to prove that every rational nymber, a/b, can be represented as a finite continued fraction or by a finite sequence.

In 1737, Euler proved that every real quadratic irrational can be represented by an infinite periodic continued fraction or by a periodic (EA) sequence development. The converse was proved by Lagrange in 1770. Of course, if the number is not a quadratic irrational, but is a real algebraic number of higher degree or a transcendental real number, then its development by the (EA) cannot be periodic.

In 1839, Hermite, in one of his letters to Jacobi [12.], challenged Jacobi to find an algorithm to develop irrationals of any degree into periodic sequences. But it was only after thirty years of frustration that Jacobi in 1869 extended (EA) methods to successfully represent some cubic irrationals by means of simple continued fraction.

Then in 1907, Perron generalized the work of Jacobi. This generalization is known as the Jacobi-Perron Algorithm (JPA). In its general form, as defined by Jacobi for $n = 3$ and by Perron for any $n \geq 2$, an application of the (JPA) starts with the definition of an initial vector, $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-1}^{(0)}) \in R^{n-1}$, $n \geq 2$ the components of which are algebraic numbers. By use of the greatest integer function a "companion vector"

$$b^{(0)} = (b_1^{(0)}, b_2^{(0)}, \dots, b_{n-1}^{(0)}) \in R^{n-1} \qquad \text{with}$$

$$b_i^{(0)} = [a_i^{(0)}], \ (i = 1,2,\dots,n-1)$$

is defined. A recursive transformation

$$a^{(v+1)} = (a_1^{(v)} - b_1^{(v)})^{-1} (a_2^{(v)} - b_2^{(v)}, \dots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1)$$

is constructed and applied to these vectors. Then the sequence

$\{a^{(v)}\}$, $v = 0,1,\dots$; is called JPA.

For good choices of the starting vector $a^{(0)}$ and for the transformation, the iteration of the transformation becomes periodic, that is the transformation cycles around a finite set of vectors. In this instance the (JPA) is said to be periodic, and the results lead to the (JPA) periodic representation of higher degree irrationals. The difficulties associated with this work are many. Jacobi's results were confined to a few numerical examples in a cubic field, where Jacobi exhibited periodic developments for $\sqrt[3]{2}$, $\sqrt[3]{3}$ and $\sqrt[3]{5}$. Perron generalized the method to apply to irrationals of any degree, but since the choices of starting vector and transformation are difficult to make, he was also limited to a few periodic developments of higher degree irrationals. Those results were to prove an Euler direction for higher degree irrationals. Perron was more successful in showing that if a development is periodic then the components of the initial vector are algebraic numbers. This latter result was general, with this proving completely Lagrange direction for higher degree irrationals.

Advances were slow and difficult, but in 1873 Bachman proved results for other cubic irrationals using the (JPA); results that were accompanied by many restrictions. With this work on Hermite's Problem progress came to a halt, because of the failure of the (JPA) to produce new numerical results, that is, additional cases in which the transformation becomes periodic were not achieved. Perron and all others recognized that the usual choices for starting vector were too limited. No further progress occurred on these problems until Hasse and Bernstein turned their attention to them in 1965, and made a broader approach to the periodicity problem associated with the (JPA). Hasse and Bernstein started with an algebraic extension of the rational numbers, $Q(w)$, where w takes form $w = \sqrt[n]{D^n+d}$ with $P(x) = \left(\prod_{i=1}^{n} (x^n - D_i^n) - d\right)$, $d \in Z$, $D_i \in N$ and $d|D$.

$$a^{(0)} = ((w-D_1)(w-D_2)\cdots(w-D_{n-1}),\ldots,(w-D_1)(w-D_2),(w-D_2))$$

with $b^{(0)} = a^{(0)}(D_1)$. They showed [9],[10], that certain restrictions on D and d led to a (JPA) that was purely periodic (that is that the length of the preperiod is zero). For $d > 0$ they proved that (JPA) of $a^{(0)}$ is purely periodic when $D \geq (n-2)d$, $d|D$ and $n \geq 3$. For $d < 0$ the sequence is also purely periodic when $D \geq 2(n-1)d$, $d|D$ and $n \geq 3$. With these conditions, the length of the period is $n(n-1)$.

For this approach the periodicity remains an open problem since there are bounds on D and the restriction $d|D$ must hold. For example no periodicity for $w = \sqrt[5]{12^5+6}$ can be proved under (HBA) restrictions since $12 \not\geq (5-2)6 = 18$.

The Hasse and Bernstein results were limited by their choices of w as real numbers. It should be noted that Hasse and Bernstein were not interested in Hermite's problem in spite of the fact that their results had a strong relation to that problem. Specifically, they did not realize that the periodicity of the algorithm leads to a solution of Hermite's Problem for some real algebraic number w.

In 1980, Baica defined a modification of the (JPA) that used the Hasse and Bernstein initial vector, but was not restricted to the real numbers. For the first time the complex numbers were considered. The only differences in the definitions stated alone are that the D_i's are now complex numbers. An immediate consequence of this extension is that the bounds on D in the (HBA) are now eliminated and only the divisibility condition, $d|D$, remains. Returning to the example cited above, it can now be seen that $w = \sqrt[5]{12^5+6}$ has a periodic development, only $6|12$ is required. Baica named her Algorithm, the Algorithm for Complex Numbers (ACF) and later she named (ACF) to be the General Euclidean Algorithm (BGEA). All of the previous results of the (JPA) and all of the (HBA) results are consequences of (BGEA). In conclusion, Baica proved that all of the real numbers, and, for the first time, all the complex numbers of the form w with $d|D$ have a periodic (BGEA) sequence development. This is the solution of Hermite's problem. From the fact that (BGEA) is not always periodic for $n \geq 3$ it follows that not all higher degree irrationals have a periodic (BGEA) sequence development.

All of the quadratic irrationals do have a (BGEA) periodic sequence development since, for $n = 2$, (BGEA) becomes (EA). This is the justification for naming (ACF) as (BGEA). (EA) and (BGEA) are always periodic for $n = 2$, but (BGEA) is not always periodic for $n \geq 3$. When $n \geq 3$ the restriction $d|D$ can not be removed in proofs of the periodicity of (BGEA).

1. PREVIOUS RESULTS FROM THE PERIODICITY OF (EA). (BGEA) for $n = 2$

(1.1) Construction with the ruler and the compass of the quadratic irrationals on the real line.

(1.2) Every real quadratic irrational can be represented by an infinite periodic continued fraction (EA) development. This is known as Euler-Lagrange Theorem.

(1.3) Explicit solutions of PELL's equations
$$x^2 - ay^2 = \pm 1 \text{ and } \pm 4.$$

(1.4) The problem to find the GALOIS' group of multiplicative units in quadratic algebraic number fields was completely solved when Pell's equation was completely solved, (Dirichlet's problem for $n = 2$).

(1.5) The existence of an algorithm to find the square root of a number to approximate the quadratic irrationals.

(1.6) The existence of a formula to solve $ax^2 + bx + c = 0$ called the quadratic equation formula.

(1.7) The determination of Pythagorean triples which leads to the integral solution of $x^2 + y^2 = z^2$.

2. FAMOUS PROBLEMS FOR $n > 2$

(2.1) To prove the one to one correspondence between the real numbers and the oriented straight line.

(2.2) Hermite's problem to find a periodic algorithmic development for higher degree irrationals.

(2.3) Solutions for higher degree diophantine equations.

(2.4) The problem to find the Galois' group of multiplicative units in higher degree algebraic number fields (Dirichlet's problem for any n).

(2.5) The existence of an algorithm to approximate higher degree irrationals once that Hilbert Completeness Axiom was accepted.

(2.6) To find relations between roots and coefficients for higher degree polynomials, as related to Galois' theory of polynomials.

(2.7) Ability to prove Fermat's last theorem to show that no integer solutions for $x^n + y^n = z^n$ for $n > 2$.

All of those open questions for $n > 2$ caused Hilbert to ask for the invention of a universal algorithm as powerful as (EA) for $n = 2$ in order to solve all of the previously mentioned problems in higher dimensions from the periodicity of this universal algorithm. Let call this demand of Hilbert as Hilbert's Universal Algorithm Periodicity Problem (HUAPP).

3. SOME PREVIOUS RESULTS OF THE AUTHOR FROM THE PERIODICITY OF (BGEA)

In [1] the author concluded the work of Jacobi, Perron, Hasse and Bernstein, and found (BGEA). Baica proved that her (BGEA) is periodic if $d|D$ for $w = \sqrt[n]{D^n+d}$, $n > 2$, $D \in N$, $d \in z$. If $d \nmid D$ (BGEA) fails to be periodic.

In [2] the author gave the justification of the need of Hilbert's Completeness Axiom to prove (2.1) the one to one correspondence between the real numbers and the oriented straight line, and gave the (BGEA) approximation of higher degree irrationals and proved (2.5). In [3] the author proved part of Hermite's problem of (2.2). In [1] the author used the periodicity of (BGEA) to prove a theorem similar to Hasse and Bernstein [9] to find Galois' multiplicative group of fundamental units in higher degree algebraic numbers fields. In [1], [5], [6], [7], the author uses the periodicity of a common algorithm (BGEA) to provide solution for (2.4) that satisfy Hilbert's demand for a universal algorithm to disclosure the existent units in algebraic number fields of higher degree. (2.3) is related to (2.4), that is to find solutions of higher degree diophantine equations is related with the units in algebraic number fields and in [4] the author provides solutions for some very complicated diophantine equations.

All of these problems in higher dimensions fail to be solved when (BGEA) fails to be periodic. This solves completely Hermite's problem also. Recall that Jacobi and Perron initially constructed their algorithm over the real numbers as a generalization of (EA), where a modified version of (EA) becomes n = 2 in (JPA). (BGEA) is a generalization of (JPA)and (HBA) over complex numbers and n = 2 in (BGEA) becomes (EA). (BGEA) x in its limitation gives the proof of all problems mentioned above.

4. THE STATEMENT OF THE PROBLEM

Hasse once stated: "The end of the 20th century will bring the solution for Fermat's Last Theorem (FLT), and the solution will come from the use of Number Theoretical tools, as Fermat had intended."

We know that we can construct as many Geometrics as we like, we start with objects and axioms and build up proofs using Logic. Each Geometry has its corresponding associate Algebra. The original (FLT) is stated in Euclidean Variety (EV) and its corresponding Euclidean Geometry (EG) where we use Elementary Number Theory (ENT) as its associate Algebra. Later we used the classical Algebra, knowing that (EG) is the group of the five transformations, where no transformation is the zero transformation under the composition function as the group operation. All those Geometries (Euclidean, Many Projective, Spherical, Parabolical and Elliptic) do not report to each other, but they all report to the Topology. This is the reason that if we prove something in Elliptic Geometry or Elliptic Variety with its corresponding Algebra it may not be the same thing as in other Geometries. We will prove (FLT) in the (EG) or (EV), which is closer with Fermat's heart. (FLT) has to be proved by functions (transformations) not by Elliptic curves. (BGEA) is the TOOL that solves (FLT) in (EV), as it supposed to be. We used the very known transformation function which is greatest integer function as in (EA) or (JPA) or the evaluation function as in (HBA) to construct (BGEA) over the complex numbers.

5. THE PROOF OF FERMAT'S LAST THEOREM (FLT).

(FLT)

There do not exist positive integers x, y, z, n such that $x^n + y^n = z^n$; if $n \geq 3$.

THEOREM

(BGEA) not always periodic when $n \geq 3$ implies (FLT).

Proof

It is known that in quadratics $x^2 + y^2 = z^2$ has integral solutions and this is an immediate consequence of the fact that (EA) is periodic. We know that Euclidean Algorithm (EA), Jacobi Algorithm (JA), Perron Algorithm (PA), Jacobi-Perron Algorithm (JPA) and Hasse-Bernstein Algorithm (HBA) developed only real numbers of the form $w = \sqrt[n]{D^n + d}$ and the (HBA) was the closest algorithm over the reals to the General Euclidean Algorithm (GEA). (HBA) is periodic when $d > 0$, $D \geq (n-2)d$, $d|D$ and when $d < 0$, $D \geq 2(n-1)d$, $d|D$. Baica developed (HBA) for the first time over the complex numbers and that eliminated the restrictions on D and only $d|D$ remains in order to prove periodic. We named this Algorithm over complex numbers (BGEA) since no other larger numbers in range exists to have complex numbers as a subset as $R \subset C$ to make another extension. Then, we realized that (BGEA) is an explicit form of Hilbert's demanded universal algorithm from whose periodicity to solve all of the open problems in n dimensions, which are solved in quadratics ($n = 2$) from the periodicity of (EA) under the form of continued fractions. Logicians proved that Hilbert's dreamed periodic Algorithm does not exist. We proved exactly the same result, providing Mathematics with an explicit (BGEA) which is periodic for any higher degree algebraic number $w = \sqrt[n]{D^n+d}$ if $d|D$. Putting those two together it is true that if $d \nmid D$, (BGEA) is not periodic, since otherwise it will contradict (HUPP). (BGEA) is of the same cut or protogype as (EA) for starting vectors with real numbers as components which are higher degree irrationals.

That (BGEA) is the (GEA) is no doubt, under its much less powerful form was used by great mathematicians to approach similar open questions in n dimensions which were proved in quadratics from the periodicity of (EA). Therefore, if $n \geq 3$ there do not exist positive integers x, y, z, n such that $x^n + y^n = z^n$, since (BGEA) is not always periodic for $n \geq 3$, and there exist positive integers x, y, z such that $x^2 + y^2 = z^2$ since for $n = 2$ (BGEA) for reals becomes (EA) and (EA) is periodic for any quadratic. If (BGEA) would be periodic for any n, then $x^n + y^n = z^n$ would have integral solutions but this will contradict (HUPP). Therefore (BGEA) not periodic for $n \geq 3$ if $d \nmid D$ implies Fermat.

Note

(EA) is $n = 2$ for (BGEA). That is the reason that because of its periodicity many problems are solved in quadratics ($n = 2$). The degree of the irrational (quadratic) is related with the degree of the equation $x^2 + y^2 = z^2$ having integral solutions because any quadratic irrational w will make (EA) periodic. Likewise the degree n of the irrational w in (BGEA) is related with the degree n in (FLT) $x^n + y^n = z^n$. The quantity under the n degree radical D^n+d is related with the proof of the periodicity of (BGEA) for that corresponding $w = \sqrt[n]{k}$ where k can be written always as $k = D^n+D$. The condition $d \mid D$ is required to prove (BGEA) of w to be periodic as it is one of the many other conditions to prove periodic (HBA) for the same w. No hundreds pages proof is needed, the Euclidean Model is explained by the History of Mathematics and it is well known by most mathematicians, it is not needed to explain it from the scratch. Also, (BGEA),

[1] is published. All of those great Mathematicians from the History of Mathematics, starting with Euclid, helped me to prove (FLT) in Euclidean Model or (EV). (BGEA) proves all of the other open questions in higher degrees up to its periodicity, $d|D$ is required to prove periodic, and it <u>cannot</u> be eliminated. This is the key to proving (FLT). The proof of (FLT) is the work of Euclid, Jacobi, Perron, Gauss, Euler, Hermits, Hilbert, Dirichlet, Hasse, Bernstein and Baica put together. All of those great mathematicians before me ultimately were looking to solve (FLT) and historically they paved the way for me to finish the final step in its proof. The solution of (FLT) is the evolutionary development of the algorithms of Jacobi, Perron, Hasse-Bernstein, and Baica.

<u>NOTE</u>

Jacobi and Perron were interested to develop an Algorithm from whose periodicity to solve Hermite's Problem. Hasse-Bernstein were interested to solve Dirichlet's problem. The application of units cannot be sufficiently prized. Gauss himself used it to prove the truth of Fermat's conjecture in the cubic case by using units in the quadratic algebraic field $Q(\sqrt{-3})$ and Kummer in his effort to solve (FLT) took refuge in the units of cyclotomic field. London and Finkelstein [14] have written a whole book yielding information through the theory of units about the famous Mordell equation.

(BGEA) now solves Dirichlet's problem completely, satisfying Hilbert's demand and the application of units will demonstrate its dominating strength.

In conclusion (BGEA) is a very powerful algorithm when it becomes periodic. The proof of Fermat's last theorem is the conclusion of the results in all Author's papers over the years.

The (BGEA) will dominate mathematics for higher dimension fields over the years to come, exactly as (EA) dominated mathematics for quadratic fields for so many years in the past.

6. MORE RESULTS FROM THE PERIODICITY OF (BGEA)

The applications of (BGEA) do not stop here. In many other published papers I have extended the application of the periodity of (BGEA) for solutions of very complicated diophantine equations. I have developed very complicated combinatorial identities and recently, I used it to find the sums of some infinite series. For the first time I emphasized the importance of the periodicity of an algorithm as a tool to prove something in Mathematics.

BIBLIOGRAPHY

[1] M. BAICA, An Algorithm in a complex field (ACF) and its applications to the calculation of units, Pacific J. Math. (1) 110 (1984) 21-40.

[2] M. BAICA, Approximation of irrationals, Internat. J. Math. Sci. (2) 8 (1985) 303-320.

[3] M. BAICA, Hermite's Problem from the periodicity of (ACF) Algorithm, Bull. Numb. Theory, vol. XIV (1990) p. 57-67.

[4] M. BAICA, Diophantine equations and identities, Internat. J. Math. and Math. Sci. vol. 8, no. 4(1985) 755-777.

[5] M. BAICA, More units from the periodicity of an Algorithm, Bull. Numb. Theory. vol. XII (1988), p. 81-89.

[6] M. BAICA, Halter-Koch units from the periodicity of (ACF) Algorithm. Bull. Numb. Theory, vol. XIII (1989), p. 73-79.

[7] M. BAICA, Hilbert's demand for the Disclosure of units in algebraic number fields, Accepted for publication.

[8] M. BAICA, Pythagorean Triangles of equal areas, Internat. J. Math. Sci. (11) 4 (1988) 769-780.

[9] L. BERNSTEIN AND H. HASSE, Einheitenberechnung mittels des Jacobi-Perronschen Algorithmus. J. Reine. Angew. Math. 218 (1965), 51-69.

[10] L. BERNSTEIN, The Jacobi-Perron Algorithm, its Theory and Applications, Springer, Berlin-Heidelberg-New York, Zeit. Notes. Math. Vol. 207, 1971.

[11] L. BERNSTEIN, Representation of $\sqrt[n]{D^n-d}$ as a periodic continued fraction by Jacobi's Algorithm, Math. Nachr. 29 (1965) 179-200.

[12] C. H. HERMITE, Letter to C. G. Jacobi, J. f.d. Reine. Angew. Math. 40 (1939) 286.

[13] C. G. J. JACOBI, Allgemeine Theorie der kettenbruchaehnlichen Algorithmen, in velchen jede Zahl aus drei vorhergehenden gebildet wird, J.f.d. reine angew. Math. 69 (1969(, 29-64.

[14] H. LONDON and R. FINKELSTEIN, On Mordell's Equation. Bowling Green State University Press, Bowling Green 1973.

[15] O. PERRON, Grundlagen füer eine Theorie des Jacobischen Kettenbruchalgorithmus, Math. Ann. 64 (1907), 1-76.

[16] O. PERRON, Ein neues Konvergenzkriterium für Jacobi-Ketten 2. Ordnung Arch. Math. Phys. (Reihe 3) 17 (1911), 204-211.

Dr. Malvina Baica
Department of Mathematics and Comp. Science
University of Wisconsin
Whitewater, Wisconsin 53190 U.S.A.