19

BULETINUL ŞTIINŢIFIC
AL
UNIVERSITĂŢII "POLITEHNICA" DIN TIMIŞOARA

TOM 42 (56), 2 MATEMATICĂ-FIZICĂ 1997

# BAICA'S GENERAL EUCLIDEAN ALGORITHM RESTRICTED PERIODICITY AN N-DIMENSIONAL EQUIVALENT FOR EULER-LAGRANGE THEOREM FROM QUADRATICS

Malvina BAICA

ABSTRACT. The Euclidean Algorithm (EA) is a very powerful tool in Quadratic Euclidean Space ($E^2$) because it is periodic as proven by the Euler-Lagrange Theorem (ELT). Consequently, many problems which remained unsolved in n-dimensional Euclidean Space ($E^n$) are proved in ($E^2$). This led Hilbert to ask mathematicians for the invention of an universal algorithm from whose periodicity to solve everything in ($E^n$) known as Hilbert's "Zahlbericht". Baica's General Euclidean Algorithm (BGEA) is an explicit form of Hilbert's demanded algorithm and in proving its restricted periodicity it becomes an n-dimesional equivalent in ($E^n$) for the Euler-Lagrange Theorem from quadratics.



| **Key words and Phrases:** | **Abbreviation:** |
|---|---|
| Euclidean Algorithm | (EA) |
| Baica's General Euclidean Algorithm | (BGEA) |
| Euler Lagrange Theorem | (ELT) |
| Hilbert's Universal Algorithm Periodicity Problem | (HUAPP) |
| Quadratic Euclidean Space | ($E^2$) |
| N-Dimensional Euclidean Space | ($E^n$) |



**1. Introduction.** Problems are often solved in mathematical models. We can generate as many mathematical models as we please. The elements are declared in order to determine the axioms and the definition of the model. Using the logic consistent with the axioms and definitions we can build any mathematical model required to solve mathematical problems in that corresponding model. Likewise, we construct numerous geometrical models or geometrys, but only one is the Euclidean geometry, all of the others are called non-Euclidean geometries. The classical Euclidean geometry is ($E^n$) in wich n=2 or ($E^2$) is known as the Euclidean geometry in quadratics.

AMS (1991) Subject Classification: 11Y40, 11A05



To every
converse is not tr
number theory (
everything in any
*System.* If we wa
conditions of the
theorem becomes
**2. Stateme**
The Eucli
number theory in
Euclidean Algorit
number theory in
This is do
its periodicity, as
quadratics ($E^2$) as
the periodicity of
restricted periodic
**3. Solution**
In 1737,
by an infinite pe
Algorithm sequen
course, if the num
higher degree or a
Algorithm can n
Theorem and pr
continued fraction
Algorithm.
In 1839,
an algorithm to de
only after thirty
Algorithm method
algorithm.

To every geometry there is a corresponding specific algebra (although the converse is not true). The algebra corresponding to a specific geometry is called the number theory (or arithmetic) of that geometry. The tool that proves almost everything in any such number theory corresponding to a geometry is called the *Euler System*. If we want to prove a theorem in that number theory which satisfies the conditions of the Euler System (and the implementation is done right) then this theorem becomes a direct consequence of the Euler System which itself is a theorem.

**2.Statement of the problem**

The Euclidean Algorithm is the corresponding Euler System in the classical number theory in Euclidean quadratics ($E^2$) and by generalization Baica's General Euclidean Algorithm (BGEA) is the corresponding Euler System in the algebraic number theory in Euclideann-dimensional ($E^n$).

This is done by proving that (BGEA) restricted periodicity does prove up to its periodicity, as consequences, all the problems in ($E^n$) which were proven in quadratics ($E^2$) as consequences of the Euler-Lagrange Theorem (ELT) which prove the periodicity of the Euclidean Algorithm. Baica's General Euclidean Algorithm restricted periodicity is the n-dimensional equivalent of (ELT) from quadratics.

**3.Solution of the problem**

In 1737, Euler proved that every real quadratic irrational can be represented by an infinite periodic continued fraction (p.s.c.f.) or by a periodic Euclidean Algorithm sequence development. The converse was proven by Lagrange in 1770. Of course, if the number is not a quadratic irrational, but is a real algebraic number of higher degree or a transcendental real number, then its development by the Euclidean Algorithm can not be periodic. These proofs are known as the Euler-Lagrange Theorem and proves the periodicity of the Euclidean Algorithm using simple continued fractions (p.s.c.f.) which is another interpretation of the Euclidean Algorithm.

In 1839, Hermite [4], in one of his letters to Jacobi, challenged Jacobi to find an algorithm to develop irrationals of any degree into periodic sequences. But it was only after thirty years of frustration than Jacobi, in 1869, extended Euclidean Algorithm methods to succesfully represent some cubic irrationals by means of his algorithm.

Then in 1907, Perron [5,6] generalized the work of Jacobi. This generalization was named by Hasse and Bernstein as the Jacobi-Perron Algorithm (JPA). They encountered many difficulties associated with their work to prove periodicity of their algorithm. Jacobi's results were confined to a few numerical examples in a cubic field. Perron generalized the method to apply to irrationals of any degree $n$, but since the choises of a starting vector and transformation in his algorithm are difficult to make, he was also limited to a few periodic developments of higher degree irrationals. Those results were to prove a Euler direction for higher degree irrationals. Perron was more successful in showing that if a development is periodic then the components of the initial vector are algebraic numbers. This latter result was general, with this proof completing Lagrange direction for higher degree irrationals.

Advances were slow end difficult, but in 1873 Bachman proved results for other cubic irrationals using the Jacobi-Perron Algorithm; results that were accompanied by many restrictions. With this work on Hermite's problem progress came to a halt because of the failure of the Jacobi-Perron Algorithm to produce new numerical results.

No further progress occured on these problems until Hasse and Bernstein [3] turned their attention to them in 1965 and made a broad approach to the periodicity problem associated with the Jacobi-Perron Algorithm. Hasse end Bernstein started with an algebraic extension of the rational number $Q(w)$, where $w$ takes the form $w=\sqrt[n]{D^n+d}$ and using a modification of the Jacobi-Perron Algorithm, they showed that for $d>0$ , $D\geq(n-2)d$ , $d/D$ , $n\geq3$ , and for $d<0$ , $D\geq2(n-1)d$ , $d/D$ , $n\geq3$ , their algorithm (HBA) is purely periodic and the lenght of the period is $n(n-1)$.

For this approach the periodicity (Euler direction) remains an open problem since there are bounds on D and the restriction d/D must hold for $n\geq3$ . For example no periodicity for $w=\sqrt[5]{12^5+6}$ can be proved under (HBA) since the bound on D, 12 is not $\geq(5-2)6=18$. The Hasse and Bernstein results were limited by their choices of $w$ as real numbers. It should be noted that Hasse and Bernstein were not interested in Hermite's problem in spite of the fact that their result had a strong relation to thatproblem. Specifically, they didnot realize that the periodicity of the algorithm leads to a solution of Hermite's problem for some real algebraic number $w$ and as such in



providing the Euler
1980 [2], Baica defin
Hasse and Bernstein i
*For the first t*
in the definitions st
immediate conseque
now eliminated and
example cited above
Euclidean Algorithm
her algorithm, The
proved that in the p
Euler's direction in
from quadratics.

In [1] Baica
that she proved the
(BGEA) complete
completely. This la
and it is the evo
Hasse-Bernstein, an

In Baica's
Algorithm, $n=3$ bec
modification for re
Baica's General
Algorithm is the
Euclidean Algorith
Algorithm was pro
irrational and som
Perron as numeri
algorithm) have p
$n\geq3$ which makes
(BGEA) algorithm

In this ca
Algorithm periodi

providing the Euler direction in the Euler-Lagrange Theorem for n-dimension. In 1980 [2], Baica defined a modification of the Jacobi-Perron Algorithm that used the Hasse and Bernstein initial vector, but was not restricted to the real numbers.

*For the first time* the complex numbers were considered. The *only* differences in the definitions stated alone are that the Ds are now complex numbers. An immediate consequence of the extension is that the bounds on D in the (HBA) are now eliminated and only the divisibility condition remains for $n \geq 3$ . Returning to the example cited above, it can now be seen that $w=\sqrt[5]{12^5+6}$ has a Baica's General Euclidean Algorithm periodic development, since only 6/12 is required. Baica named her algorithm, The Algorithm for Complex Numbers (ACF). At that time Baica proved that in the periodicity of her algorithm d/D is a necessary condition in proving Euler's direction in the n-dimensional equivalent of the Euler-Lagrange Theorem from quadratics.

In [1] Baica proved that d/D for $n \geq 3$ is also a sufficient condition, and with that she proved the restricted periodicity of her Baica's General Euclidean Algorithm (BGEA) completely and as such she proved Euler direction for n-dimension completely. This last proof makes (BGEA) the *only* General Euclidean Algorithm and it is the evolutionary development of the algorithms of Jacobi, Perron, Hasse-Bernstein, and Baica.

In Baica's General Euclidean Algorithm, $n=2$ becomes Euclidean Algorithm, $n=3$ becomes (JA), for any $n \geq 3$ of any real number is Perron (PA), (JPA) modification for reals is (HBA) and (HBA) extension over the complex numbers is Baica's General Euclidean Algorithm. Ultimately, Baica's General Euclidean Algorithm is the General Euclidean Algorithm. Only for $n=2$ , Baica's General Euclidean Algorithm coincide with (p.s.c.f.) since the periodicity of the Euclidean Algorithm was proved by Euler and Lagrange using (s.c.f.). Therefore every quadratic irrational and some limited number of higher degree irrationals (used by Jacobi and Perron as numerical examples in their attempt to prove the periodicity of their algorithm) have periodic (s.c.f.) development. All of the other irrationals of degree $n \geq 3$ which makes Baica's General Euclidean Algorithm periodic have a periodic (BGEA) algorithmic development.

In this case Hermite's problem is to find a Baica's General Euclidean Algorithm periodic algorithmic development for $n \geq 3$ degree irrationals. In proving

(BGEA) periodic for $w=\sqrt[n]{D^n+d}$ with $d/D$ for $n\geq3$ means that every $w$ which makes Baica's General Euclidean Algorithm periodic will have a (BGEA) periodic algorithm development. This Baica's General Euclidean Algorithm development is not by (s.c.f.) and proves the Euler direction in the n-dimensional equivalent of the Euler Lagrange Theorem from quadratics.

As it can be seen, it is not only the beginning Euclidean Algorithm and the end Baica's General Euclidean Algorithm is so much else in between then only the gap that makes them apart for more then 2000 years. It is all of the genial work of all of those great mathematicians before me, who historically paved the way for me to finish this final step and give the mathematics this very powerful tool which is the General Euclidean Algorithm. In proving its restricted periodicity, (Euler direction) Baica proved that Baica's General Euclidean Algorithm restricted periodicity becomes a theorem which is the n-dimensional equivalent of the Euler-Lagrange Theorem from quadratics, Lagrange direction being proved by Perron in 1907. Therefore, this very well known and powerful tool, the Euclidean Algorithm is the Euler System equivalent in the number theory of $(E^2)$, Baica's General Euclidean Algorithm is the Euler System equivalent in the algebraic number theory of $(E^n)$ exactly as Euler System is the tool in the number theory which corresponds to a specific geometry.

In chapter 1 of [1], Baica identified some of the problems in quadratics wich are direct consequences of the always periodicity of the Euclidean Algorithm or of the Euler-Lagrange Theorem. In chapter 2 of [1], Baica identified all the problems which were open questions in n-dimensions before the invention of Baica's General Euclidean Algorithm. In chapter 3 of [1], Baica identified all her publications in which she partially proved up to (GEA) necessary condition restricted periodicity, all those open problems in chapter 2 of [1].

In [1] Baica proves that d/D is also a sufficient condition in proving (GEA) restricted periodicity making (GEA) to be Baica's General Euclidean Algorithm (BGEA), and therefore completing the proof for the n-dimensional equivalent of Euler-Lagrange theorem from quadratics. That shows that all of the problems identified in chapter 2 of [1] as open questions are now consequences of Baica's General Euclidean Algorithm restricted periodicity by generalization as well as all the problems identified in chapter 1 of [1] are consequences of the Euler LaGrange Theorem. In further papers the author will revise all of the results in chapter 3 of [1]



using this new approa
powerful algorithm
analogue in $(E^n)$.

The Baica's
higher dimensions (
dominated mathema
together the work o
beginning with Eucl
its restricted periodi
those great mathem
prove its periodicity
together, their dre
Euclidean Algorith

REFERENCES

[1] M. BAICA , E
Last Theorem , N
(1995) 120-134.
[2] M. BAICA ,
calculation of unit
[3] L. BERNST
Perronschen Algo
[4] C. H. HERM
[5] O. PERR
Kettenbruchalgor
[6] O. PERRON
Arch. Math. Phy

Malvina
Department of
The University o
Whitewater , W

using this new approach. In conclusion Baica's General Euclidean Algorithm is a very powerful algorithm when it becomes periodic, and is the Euler System Euclidean analogue in ($E^n$).

The Baica's General Euclidean Algorithm will dominate mathematics for higher dimensions ($E^n$) over the years to come, exactly as the Euclidean Algorithm dominated mathematics for quadratics ($E^2$) for so many years in the past. It put together the work of great mathematicians during of the entire history of mathematics beginning with Euclid. They paved the way for me to finish the final step in proving its restricted periodicity and as such to invent the General Euclidean Algorithm. All of those great mathematicians aimed to produce the General Euclidean Algorithm and to prove its periodicity some time in their life, and with their genial work which I put together, their dreams becomes a realisation and now we have Baica's General Euclidean Algorithm (BGEA) to be this General Euclidean Algorithm.

## REFERENCES


[1] M. BAICA , Baica's General Euclidean Algorithm and the solution of Fermat's Last Theorem , Notes on Number Theory and Discrete Mathematics , Vol. 1 , No.3, (1995) 120-134.

[2] M. BAICA , An Algorithm in a complex field (ACF) and its applications to the calculation of units , Pacific J. Math. (1) 110 (1984) 21-40.

[3] L. BERNSTEIN and H. HASSE , Einheitenberechnung mittles des Jacobi-Perronschen Algorithmus , J. Rein , Agnew. Math. 218 (1965) 51-69.

[4] C. H. HERMITE , Letter to C.G. Jacobi , J.f.d. Reine. Agnew 40 (1939) 286.

[5] O. PERRON , Grundlagen fuer eine Theorie des Jacobisschen Kettenbruchalgorithmus , Math. Ann. 64 (1907) 1-76.

[6] O. PERRON , Ein Neues Konvergenzkriterium fur Jacobi-Ketten 2.Ordnung Arch. Math. Phys. (Reine 3) 17 (1911) 204 -211.



Malvina Baica
Department of Mathematics and Computer Science
The University of Wisconsin-Whitewater
Whitewater , Wisconsin 53190