

THE EULER SYSTEM FOR THE  
ALGEBRAIC NUMBER THEORY

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**Abstract**

In the present paper the author shows that her General Euclidean Algorithm (BGEA) [1,10] is the Euler System of the Algebraic Number Theory.

**Key words and Phrases:**

BAICA'S ALGORITHM IN A COMPLEX FIELD

JACOBI - PERRON ALGORITHM

(ACF) ALSO NAMED BAICA'S GENERAL EUCLIDEAN ALGORITHM

HASSE - BERNSTEIN ALGORITHM

**Abbreviation:**

(ACF)

(JPA)

(BGEA)

(HBA)

**1.Introduction.** The foundation of this paper is an Algorithm, and we shall therefore give a short historical survey of its development. It all started with the well known Euclidean Algorithm (EA)

known to Euclid more than 2000 years ago. For instance,

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using this Algorithm, it is easy to prove that every rational number can be represented as a finite simple continued fraction. Euler (1737) and Lagrange (1770) proved that every real quadratic irrational is represented by an infinite periodic continued fraction and every infinite periodic continued fraction represents a real quadratic irrational. This is known as Euler – Lagrange Theorem (ELT) for quadratics, and it proves the periodicity of the Euclidean Algorithm. In 1839, Hermite [18], in one of his letters to Jacobi, [19], challenged Jacobi to find an Algorithm to develop irrationals of any degree into periodic sequences, a problem known as *Hermite's Problem*. Jacobi generalized the (EA) for the cubic case. This was later generalised by Perron [20] who proved the Lagrange direction in (ELT) that is, it can become Periodic only for algebraic numbers. Unfortunately, except for a few numerical examples, neither Jacobi nor Perron succeeded in proving the periodicity, which is Euler direction in (ELT) of their new algorithm called Jacobi – Perron Algorithm (JPA).

Periodicity is a very important property. For instance, in the quadratic case, it enables us to solve the incorrectly named Pellian Equation. It is an Euler equation  $x^2 - my^2 = \pm 1$  or  $\pm 4$  where  $m$  is a square free natural number. Periodicity enables us to find the fundamental unit in the quadratic fields  $Q(\sqrt{m})$ . The problem of finding the multiplicative group of units in any algebraic field  $F$  over the field  $Q$  of rationals is a difficult question known as the *Dirichlet Problem*. A breakthrough in finding explicitly stated units in a wide class of real algebraic function fields took place when Hasse and Bernstein [21] succeeded in showing that (JPA) becomes periodic for a properly chosen initial vector. From such a periodic algorithm called Hasse-Bernstein Algorithm (HBÄ) they succeeded in finding some explicit units in the corresponding fields. The most important among these are the algebraic number fields, which include the quadratic one and have the form :

A)  $Q(w)$ ,  $w^n = D^n + d$ ,  $D, d \in \mathbb{N}$ ,  $d \mid D$  and  $D \geq d(n-2)$

B)  $Q(w)$ ,  $w^n = D^n - d$ ,  $D, d \in \mathbb{N}$ ,  $d \mid D$  and  $D \geq 2d(n-1)$

For (HBA), the Euler direction in (ELT) remains an open problem since there are bounds on  $D$  and  $d \mid D$  must hold for  $n \geq 3$ .

Hasse and Bernstein were not interested in Hermite's problem, but they were interested in Dirichlet's problem. In 1980 [1], Baica defined a modification of the Jacobi - Perron algorithm (JPA) that used the Hasse and Bernstein initial vector, but was not restricted to the real numbers. For the first time, the complex number were considered. The only differences in the definitions stated alone are that the  $D_i$ 's are now complex numbers. An immediate consequence of the extension is that the bounds on  $D$  in the (HBA) are eliminated and only the divisibility condition  $d \mid D$  remains for  $n \geq 3$ . For example,  $w = \sqrt[5]{12^5 + 6}$  does not make (HBA) periodic because  $D = 12 \neq d(n-2) = 6(5-2) = 18$ , but it makes Baica's algorithm periodic. Baica named her algorithm, the Algorithm for complex numbers (ACF). At that time, Baica proved that in the periodicity of her algorithm,  $d \mid D$  is a necessary condition in proving the Euler direction for the periodicity of her algorithm.

Hasse, who was her PhD dissertation advisor, asked Baica to do something with this restriction, that is, to show either that it can be eliminated and the Fermat Last Theorem is false or it can not be eliminated. This will mean to prove that  $d \mid D$  is also the sufficient condition in proving the Euler direction for the periodicity of her algorithm. As a consequence, the Fermat Last Theorem will be true. It can be proved from its restricted periodicity.

## 2. Statement of the Problem

In the 1940's, E. Schmidt was the first to say that since we can not compute in a geometry, to every geometry is corresponding an algebra. He is the father of the Algebraic Geometry as F. Gauss is the father of the Algebraic Number Theory. The algebra corresponding to a specific geometry is called the number theory (or arithmetic) of that geometry. The tool that proves almost everything is any such number theory corresponding to a geometry is called the Euler System of that number theory. The algebraic number theory is the algebra of the  $n$  - dimensional euclidean geometry ( $E^n$ ). We will show that Baica's General Euclidean Algorithm (BGEA) is the Euler System of the algebraic number theory.

## 3. Solution of the Problem

In ch. 1 of [10] the author stated all the problems in quadratics or  $E^2$  which were completely proved from the periodicity of (EA).

In ch. 2 of [10] the author stated all the famous problems in  $E^n$  for  $n > 2$  which were generalization of the problems in ch. 1, problems that were open questions in the algebraic number theory. All of these open questions for  $n > 2$  caused Hilbert to ask for the invention of a universal algorithm as powerful as (EA) with  $n=2$  in order to solve all of the problems in ch. 2 for higher dimensions from the periodicity of this universal algorithm. This Hilbert "Zahlbericht" is known as Hilbert's 10<sup>th</sup> problem. Logicians proved that Hilbert's *Dreamed Always Periodic Algorithm* does not exist.

In ch. 3 of [10], the author identified all her publications in which she partially proved up to (ACF), [then known as (GEA)], necessary condition

restricted periodicity, all those open problems in ch. 2 of [10]. In [10] Baica used the result proved by the logicians of Hilbert's 10<sup>th</sup> problem to prove that  $d|D$  is also a sufficient condition in proving (GEA) restricted periodicity. This transforms (ACF) to Baica's General Euclidean Algorithm (BGEA). Therefore, the proof is completed for the Euler direction, achieved by proving the restricted periodicity of (BGEA). The dimension of (BGEA) algorithm is equal to the degree of the irrational which makes it restricted periodic, and  $n=2$  in (BGEA) is the original (EA) where every quadratic irrational makes (EA) periodic.

*By showing that (BGEA) with its restricted periodicity is the only algorithmic explicit proof of Hilbert's 10<sup>th</sup> problem, we will prove that (BGEA) is the Euler System (ES) for the algebraic number theory which is the number theory of the  $n$ -dimensional Euclidean Geometry ( $E^n$ ).*

The author's previous results partially solved the open problems in the algebraic number theory of  $n$ -dimensions, using only  $d|D$  as a necessary condition in proving (BGEA) restricted periodicity. When a sufficient condition for  $d|D$  was provided, it solve completely all the same problems up to (BGEA) restricted periodicity. When (BGEA) is not periodic, those problems do not have solutions. We show that the (BGEA) restricted periodic algorithm is Hilbert's dream explicit algorithm by identifying the papers and the problems which the author solved from (BGEA) restricted periodicity over the years.

The original Fibonacci numbers can be derived from the periodic expansion by the (EA) of  $\sqrt{5}$  or by a periodic continued fraction development of  $\sqrt{5}$  only.

- 1) In [2] we opened a new horizon for the generalization for the Fibonacci numbers and used (BGEA) restricted periodicity to derive  $n$ -dimensional

Fibonacci numbers where for  $n=2$  in the  $n$ -dimensional Fibonacci numbers we obtain the original Fibonacci number.

- 2) In [3] we proved that (BGEA) is also the general continued fraction algorithm and it is much more, is the only General Euclidean Algorithm.
- 3) In [4] we proved Hermite's problem.
- 4) In [1,5,6,7] we proved Dirichlet's problem. The Galois' multiplicative group of fundamental units in algebraic number fields give a complete solution to Galois' theory of polynomials, providing the factorization of higher degree polynomials. Once the factorization is known, then we can find the solutions of higher degree polynomial equations.
- 5) In [8] the author proved the need of Hilbert's completeness axiom to be introduced in logic in order to prove the one to one correspondence between the real numbers and the oriented straight line and gave the (BGEA) approximation of higher degree irrationals.
- 6) In [9] the author provides solutions for some very complicated diophantine equations.
- 7) In [14] we solved  $x^2 \pm xy + y^2 = z^2$  known as Hasse's equation explicitly, using Pell's equation. Since this is a homogeneous diophantine equation we could find a transformation to reduce this quadratic diophantine equation to Pell's equation. This diophantine equation was solved parametrically by Hasse. Hilbert stated that  $x^2 + y^2 = z^2$  has integer solutions because (EA) is always periodic and brings us to find its explicit solutions. This paper confirms Hilbert's statement using the fact that homogeneous quadratic equation can be solved explicitly from the periodicity of the (EA).
- 8) In [10,11,12] the author proved the only euclidean solution [22] of Fermat's Last Theorem (FTL).
- 9) In [13] we developed complicated combinatorial identities.

10) In [16,17] we used (BGEA) restricted periodicity to find the sums of some infinite series.

11) In [15] the author proved an  $n$ -dimensional equivalent of (ELT) in quadratics. In conclusion (BGEA) is a very powerful algorithm when it is periodic. All of these problems in higher dimensions do not have solutions when (BGEA) fails to be periodic. (BGEA) is the evolutionary development of the algorithms of Jacobi, Perron, Hasse-Bernstein and Baica. (BGEA) solved up to its restricted periodicity all the open questions in the algebraic number theory, and this proves that (BGEA) is the explicit form of Hilbert's demanded algorithm. In Hilbert's 10<sup>th</sup> problem, he was asking for the General Euclidean Algorithm (GEA) which will prove from its periodicity all the open questions in  $n$ -dimensions originally proved in quadratics from the always periodicity of the (EA). (BGEA) does this when  $d \mid D$ . With this we proved that (EA) is the Euler System (ES) in quadratics and (BGEA) is the Euler System (ES) in  $n$ -dimensions for the algebraic number theory which is the number theory of the  $n$ -dimensional Euclidean Geometry ( $E^n$ ).

The (BGEA) will dominate mathematics for higher dimensions ( $E^n$ ) over the years to come, exactly as the euclidean Algorithm dominated mathematics for quadratics ( $E^2$ ) for so many years in the past. It put together the work of great mathematicians during the entire history of mathematics beginning with Euclid and finishing with Baica and so much in between. This is so much in between helped me to produce (BGEA) the General Euclidean Algorithm which is the Euler System (ES) in  $E^n$  for the Algebraic Number Theory.

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