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# CLARIFICATIONS OF THE AUTHOR'S PREVIOUS PAPER ON GOLDBACH'S CONJECTURE

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**ABSTRACT:** In a previous paper [1] the author gave a tentative proof of Goldbach's Conjecture. The purpose of this paper is to provide additional explanations and to give a definitive version of the above mentioned work.

## 1. Introduction

In [1] we did not state the relation between the $\vartheta$ - and $\nu$-functions in the best way. However, if we do it in this new more correct way (which we shall explain later in this paper) it will not affect the final result. Also, Lemma 1 of [1] will be slightly changed. This improved paper will complement her previous paper [1].

## 2. The relations between $\vartheta$ ([*u*]) and $\nu$ (*t*).

Starting with Chebyshev's function $\vartheta([u])$:

$$(2.1) \qquad \vartheta([u]) = \sum_{p \le [u]} \log p$$

where $p$ denotes the prime numbers, we have:

$$(2.2) \qquad \Delta\vartheta([u]) = \vartheta([u]) - \vartheta([u-1]) = \begin{cases} \log p \ \text{ if } [u] = p \\ o \ \text{ in any other case} \end{cases}$$

Hence, if we form

$$(2.3) \qquad \nu(t) = \sum_{t = u_1 + u_2} \Delta\vartheta([u_1])\,\Delta\vartheta([u_2])$$

where $\nu(t)$ is the Hardy – Littlewood function

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(2.4) $$v(t) = \sum_{p_1 + p_2 = t} \log p_1 \cdot \log p_2 \ ,$$

From (2.3) we have:

(2.5) $$v(t) = \sum_{u_1 < t} \Delta \vartheta([u_1]) \, \Delta \vartheta(t - [u_1]) \ .$$

### 3. Transformation of the sum in (2.5) into an integral.

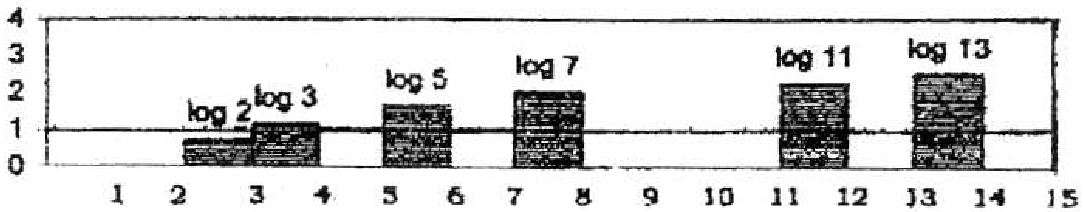
The $\Delta\ \vartheta\ ([u])$ function, when not zero, can be drawn as segments of length 1 at heights $\log p$ every time that $u = p$. More precisely, we have:

(3.1) $$\Delta\ \vartheta\ (u) = \log p$$

whenever $p \leq u < p + 1$ (otherwise it is zero)

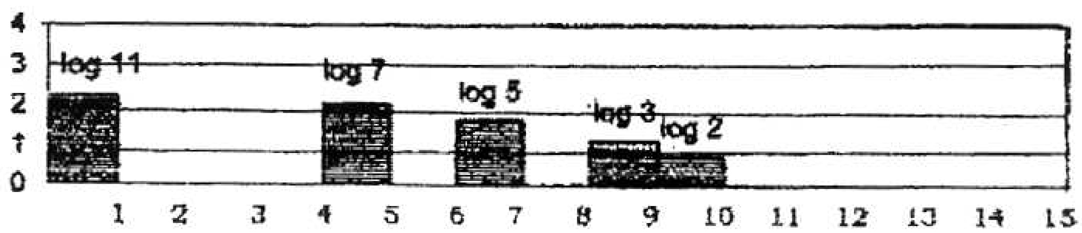
This is shown as shaded bars in fig. 1:

$\vartheta\ ([u])$ - $\vartheta\ ([u - 1])$ Fig. 1

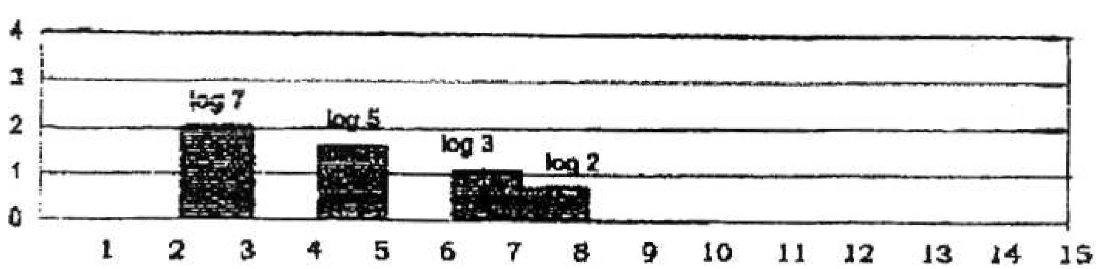



The graph of $\vartheta\ ([t - u])$ is obtained from that of $\vartheta\ ([u])$ by rotating the graph about the axis $u = t/2$. In fig. 2 and 3 are shown the cases of $y = \vartheta\ ([12 - u])$ - $\vartheta\ ([12 - u - 1])$ and $y = \vartheta\ ([10 - u])$ - $\vartheta\ ([10 - u - 1])$.

$\vartheta\ ([12 - u])$ - $\vartheta\ ([12 - u - 1])$ Fig. 2




$\vartheta\ ([10 - u])$ - $\vartheta\ ([10 - u - 1])$ Fig. 3



Hence the graph of $\vartheta([u]) - \vartheta([u-1])$ consists of unitary bars at the right of $u = p$, and that of $\vartheta([t-u]) - \vartheta([t-u-1])$ consists of unitary bars at the left of $u = p$. If we now form the product

(3.2) $$\{\vartheta([u]) - \vartheta(u-1])\}\ \{\vartheta([t-u]) - \vartheta([t-u-1])\}$$

it is zero everywhere, except at the points where $u = p_1,\ t-u = p_2,$ where it has the value

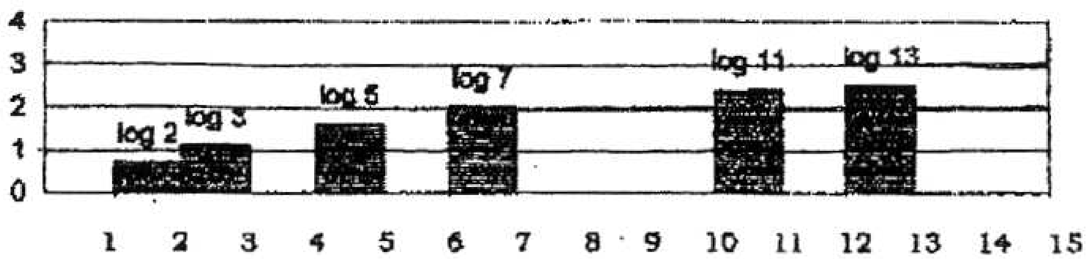
$$\log p_1 - \log p_2 .$$

It is evident from the graph the graph that if we wish to obtain this value as an area, we must displace the graph of $\Delta\,\vartheta([u])$ to the left by one unit, because then:

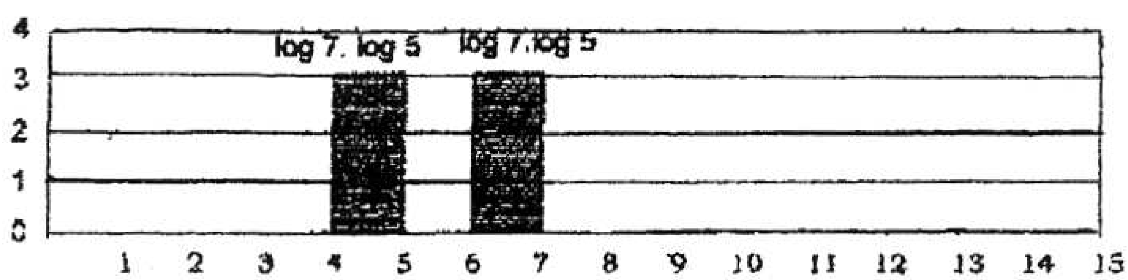
(3.3) $$\int_{p-1}^{p} \{\vartheta([u+1]) - \vartheta([u])\}\{\vartheta([t-u]) - \vartheta([t-u-1])\} = \log\, p \cdot \log(t-p)\,.$$

(The shared areas of log $p_1$ and log $p_2$ multiply between themselves). This is shown in fig. 4 and 5. The integrand in (3.3) vanishes if $u < 0$ or $u > t$.

$\vartheta([u+1]) - \vartheta([u])$ Fig. 4




$\{\vartheta([u+1]) - \vartheta([u])\}\ \{\vartheta([12-u]) - \vartheta([12-u-1])\}$ Fig. 5




Hence, we have that

(3.4) $$v(t) = \int_{0}^{t} \Delta\,\vartheta([u]+1)\,\Delta\,\vartheta([t-u])\,du\,.$$

This is nothing but the convolution of the functions that appear on the right. Consequently, if we take the Laplace transform of both sides we obtain, by known properties:

(3.5) $\quad L\{\nu(t)\} = L\{\Delta\vartheta([u]+1)\}\cdot L\{\Delta\vartheta([u])\}.$

By the second shift property of the Laplace transform we have:

$$L\{\Delta\vartheta([u]+1)\} = e^{s} L\{\Delta\vartheta([u])\}$$

so that (3.5) transforms to

(3.6) $\quad L\{\nu(t)\} = e^{s}\ L^2\ \{\Delta\vartheta([u])\}$

and also:

(3.7) $\quad L\{\nu(t-1)\} = L^2\{\Delta\vartheta([u])\}.$

This will be used in §12 later.

## 4. The evaluation of L {Δ ϑ ([u]}

According to ref. [7] we have:

(4.1) $\quad L\{\vartheta([u])\} = \dfrac{1-e^{-s}}{s}\sum_{k=0}^{\infty}\vartheta(k)e^{-ks}.$

From this it follows that

(4.2) $\quad L\{\Delta\vartheta([u])\} = \dfrac{1-e^{-s}}{s}\sum_{p}\ \log p\cdot e^{-ps}$

($p$: prime numbers) on account of (2.2).

By ref.[3], theorem 248 p. 212, we have:

(4.3) $\quad F(x) = \Sigma\log p\,x^{p} = \sum_{q=1}^{[\sqrt{n}]}\sum_{\substack{h=1\\(h,q)=1}}^{q}\dfrac{\mu(q)e^{2\pi ih/q}}{\varphi(q)(x-e^{2\pi ih/q})} + An^{\vartheta+1/4+\varepsilon};\ \left(|x| = e^{-1/n}\right).$

As a consequence of this we have that

(4.4) $\quad F(s) = \Sigma\log p\cdot e^{-ps} = \sum_{q=1}^{[\sqrt{n}]}\sum_{\substack{h=1\\(h,q)=1}}^{q}\dfrac{\mu(q)}{\varphi(q)(s+2\pi ih/q)} + An^{\vartheta+1/4+\varepsilon}.$

(see ref. [6] for a detailed calculation).

This formula is unconditional, and $\vartheta$ denotes the upper bound of the real part of the imaginary zeros of the L – series involved. (Here Re($s$) = σ = $1/n$: but this restriction does not affect our further reasoning).

Let us analyze formula (4.4). The function F(s) has a double infinitude of poles on the line σ = 0, whenever $s = -2\pi i\, h/q$ where $1 \le h \le q$ and $1 \le q < \infty$. They form a natural boundary, and the series on the right of (4.4) describes the influence of the poles with $q \le [\sqrt{n}]$, while the other term on the right accounts for the influence of the poles with $q > [\sqrt{n}]$.

It is evident that the greater the knowledge we have about the zeros, the smaller is the remainder term in (4.4). At present it is known that $\vartheta = 1 - \varepsilon$, but according to the extended Riemann Hypothesis (ERH) we have $\vartheta = ½$.

## 5. Use of (4.4)

Replacement of (4.4) in (4.2), and of (4.2) in (3.6) yields:

$$\text{(5.1)} \qquad L\{\nu(t)\} = e^{s}\left(\frac{1-e^{-s}}{s}\right)^{2}\left\{\sum_{q=1}^{[\sqrt{n}]}\sum_{h=1}^{q}\frac{\mu(q)}{\varphi(q)(s+2\pi ih/q)} + An^{\vartheta+1/4+\varepsilon}\right\}^{2}$$

or

$$\nu_m(t-1) = \frac{\nu(t-1+0)+\nu(t-1-0)}{2} = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^{2}\sum_{q}^{[\sqrt{n}]}\sum_{h=1}^{q}\frac{\mu^2(q)}{\varphi^2(q)(s+2\pi ih/q)^2}\right.$$

$$\text{(5.2)} \qquad \left(\frac{1-e^{-s}}{s}\right)^{2}\sum_{q_1=1}^{[\sqrt{n}]}\sum_{h_1}\sum_{q_2=1}^{[\sqrt{n}]}\sum_{h_2}\frac{\mu(q_1)\mu(q_2)2An^{\vartheta+1/4+\varepsilon}}{\varphi(q_1)\varphi(q_2)(s+2\pi ih_1/q_1)(s+2\pi ih_2/q_2)} +$$

$$+\left.\left(\frac{1-e^{-s}}{s}\right)^{2}An^{2\vartheta+1/2+2\varepsilon}\right\} = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^{2}\left\{\sum_{q=1}^{[\sqrt{n}]}\sum_{h}\frac{\mu^2(q)}{\varphi^2(q)(s+2\pi ih/q)^2} +\right.\right.$$

$$\left.\left.+\sum_{q_1}^{[\sqrt{n}]}\sum_{h_1}\sum_{q_2}^{[\sqrt{n}]}\sum_{h_2}\frac{\mu(q_1)\mu(q_2)2An^{\vartheta+1/4+\varepsilon}}{\varphi(q_1)\varphi(q_2)(s+2\pi ih_1/q_1)(s+2\pi ih_2/q_2)}\right\}\right\} +$$

$$+L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^{2}An^{2\vartheta+1/2+2\varepsilon}\right\} =$$

$$\text{(5.3)} \qquad L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^{2}\{T_1(s)+T_2(s)\}\right\} + T_3 .$$

For reasons that will become apparent in §10, we shall first evaluate $L^{-1}\{T_1(s)\}$ and $L^{-1}\{T_2(s)\}$.

According to the calculations performed in ref.[3] we can adopt for A the value

$$\text{(5.4)} \qquad A = 80$$

(This value however is not relevant for what follows).

## 6. Evaluation of $L^{-1}\{T_1(s)\}$

Appealing to tables we have:

$$\text{(6.1)} \qquad L^{-1}\{T_1(s)\} = L^{-1}\left\{\sum_{q=1}^{N}\sum_{h}\frac{\mu^2(q)}{\varphi^2(q)\,(s+2\pi ih/q)^2}\right\} =$$

$$= \sum_{q=1}^{N}\sum_{h=q}^{q-1}\frac{\mu^2(q)}{\varphi^2(q)}e^{-2\pi iht/qt} = \sum_{q=1}^{N}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)\,t$$

where

$$C_q(t) = \sum_{\substack{h=0\\(h,q)=1}}^{q-1} e^{-2\pi iht/q}$$

is Ramanujan's function, and $N = \left[\sqrt{n}\right]$.

According to Lemma 1 we have

$$\left|\sum_{q>N}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)\right| < d(t)e^{3\gamma}\quad\frac{(\log\log n)^2}{N}\log\log t\,.$$

Hence in (6.1) we can put

$$L^{-1}\{T_1(s)\} = \sum_{q=1}^{\infty}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)\,t +$$

$$\text{(6.2)} \qquad + \delta_1 e^{3\gamma}\;d(t)\,\frac{(\log\log N)^2}{N}\log\log t\,.$$

Now the singular series

$$\text{(6.3)} \qquad S(t) = \sum_{q=1}^{\infty}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)$$

vanishes for odd $t$, and for even t can be transformed into an infinite product (ref. [9]):

$$\text{(6.4)} \qquad S(t) = 2\prod_{p=3}^{\infty}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p/t}\frac{p-1}{p-2}t = 1.3203\prod_{p/t}\frac{p-1}{p-2}\,t = P(t)t$$

which reveals that $S(t)$ is a discontinuous function.

We have thus analyzed the influence of the infinitude of double poles in (5.2).

## 7. Evaluation of $L^{-1}\{T_2(s)\}$

According to tables we have:

$$\text{(7.1)} \qquad F_5(t) = L^{-1}\{T_2(s)\} =$$

$$= L^{-1}\left\{\sum_{q_1=1}^{N}\sum_{h_1}\sum_{q_2=2}^{N}\sum_{h_2}\frac{\mu(q_1)\mu(q_2)\,2AN^{2\vartheta+1/2+2\varepsilon}}{\varphi(q_1)\varphi(q_2)\,(s+2\pi ih_1/q_1)(s+2\pi ih_2/q_2)}\right\} =$$

$$= \sum_{q_1=1}^{N} \sum_{h_1} \sum_{q_2=2}^{N} \sum_{h_2} \frac{\mu(q_1)\mu(q_2)}{\varphi(q_1)\varphi(q_2)} \frac{\left(e^{-A_1 t} - e^{-A_2 t}\right)}{A_2 - A_1} 2AN^{2\vartheta+1/2+2\varepsilon}$$

$(A_2 = 2\pi i h_2/q_2,\ A_1 = 2\pi i h_1/q_1)$.

Taking into account Lemma 2 we have:

(7.2) $$\left|L^{-1}\{T_2(s)\}\right| = \frac{\delta_2}{4\pi} N^2 (N+1)^2\ 2AN^{\vartheta+1/2+2\varepsilon}.$$

We have thus analyzed the effect of the simple poles with $q \le N$ multiplied by the simple poles with $q > N$ or by themselves.

## 8. Evaluation of $T_3$.

We have:

(8.1) $$T_3 = An^{2\vartheta+1/2+2\varepsilon} L^{-1}\left(\frac{1-e^{-s}}{s}\right)^2.$$

According to tables (ref.[7]), we have:

(8.2) $$F(t) = L^{-1}\left(\frac{1-e^{-s}}{s}\right)^2 = \begin{cases} t & \text{if} \quad 0 < t < 1 \\ 2-t & \text{if} \quad 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}.$$

Hence:

$|F(t)| \le 1$ for every $t$, and

(8.3) $$T_3 = A_1 n^{2\vartheta + 1/2 + 2\varepsilon} = A_1 N^{4\vartheta + 1 + 4\varepsilon}$$

with $A_1 < A \le 80$.

We have thus demonstrated the influence of the simple poles with $q > N$; and no other poles exist.

## 9. Return to (5.3)

We now evaluate

(9.1) $$F_1(t) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_1(s)\right\}$$

and

(9.2) $$F_2(t) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_2(s)\right\}.$$

## 10. Evaluation of $F_1(t)$ in (9.1)

According to formula (6.2) $F_4(t) = L^{-1}\{T_1(s)\}$ is the sum of a dominant term $D(t)$:

(10.1) $$D(t) = 1.3203 \prod_{p/t} \frac{p-1}{p-2}\, t$$

which we write as:

(10.2) $$D(t) = P(t) \cdot t$$

where

(10.3) $$P(t) = 1.3203 \prod_{p/t} \frac{p-1}{p-2}$$

is a discontinuous step function of $t$, plus a remainder term $R(t)$, given by:

(10.4) $$R(t) = \delta_3 e^{3\gamma} d(t) \frac{(\log\log N)^2}{N} \log\log t$$

that contains a discontinuous factor $d(t)$, so that

(10.5) $$F_4(t) = D(t) + R(t)\,.$$

Now, the convolution theorem for the inverse Laplace transform states that if

(10.6) $$F_3(t) = L^{-1}\{f_3(s)\} \text{ and } F_4(t) = L^{-1}\{f_4(s)\}$$

then

(10.7) $$L^{-1}\{f_3(s)\, f_4(s)\} = \int_0^t F_3(u)\, F_4(t-u)\, du\,.$$

In (10.6) we now choose

$$f_3(s) = \left(\frac{1-e^{-s}}{s}\right)^2$$

so that $F_3(t)$ is the $F(t)$ of (8.2), $f_4(s) = T_1(s)$ and $F_4(t) = L^{-1}\{T_1(s)\}$ is the function in (10.5). It follows that

(10.8) $$F_1(t) = L^{-1}\{f_3(s)\ f_4(s)\} = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_1(s)\right\} =$$

$$= \int_0^t F(u)\, F_4(t-u)\, du = \int_0^1 u\, F_4(t-u)\, du + \int_0^1 (2-u) F_4(t-u)\, du\,.$$

Now

(10.9) $$\int_0^t u\, F_4(t-u)\, du = \int_0^1 u(t-u)\, du \cdot P(t-0) + \int_0^1 u\, R(t-u)\, du$$

$$= P(t-0)\left\{\frac{t}{2}-\frac{1}{3}\right\} + \delta_4\, R(t-\delta_5) \qquad 0 < \delta_4,\ \delta_5 < 1$$

and

(10.10)
$$\int_1^2 (2-u)\, F_4(t-u)\, du = \int_1^2 (2-u)\,(t-u)\, du \cdot P(t-1+0) +$$

$$+ \int_1^2 (2-u)\, R(t-u)\, du = P(t-1+0) \left\{\frac{t}{2} - \frac{2}{3}\right\} + \delta_6\, R(t-\delta_7)$$

$$1 < \delta_6,\ \delta_7 < 2.$$

Hence

(10.11)
$$F_1(t) = P(t-0) \left\{\frac{t}{2} - \frac{1}{3}\right\} + P(t-1+0) \left\{\frac{t}{2} - \frac{2}{3}\right\} + \delta_4\, R(t-\delta_5) + \delta_6\, R(t-\delta_7).$$

## 11. Evaluation of $F_2(t)$ in (9.2)

We had:
$$F_2(t) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T^2(s)\right\}$$

But according to the convolution theorem (10.6) – (10.7)

$$F_2(t) = \int_0^t F(u) \cdot L^{-1}\, \{T_2(s)\}\, du$$

Where $F(u)$ is the function of (8.2) and $L^{-1}\{T_2(s)\}$ was calculated in (7.2). We deduce:

(11.1)
$$F_2(t) = L^{-1}\{T_2(s)\} \int_0^2 F(u)\, du = L^{-1}\{T_2(s)\} =$$

$$= \frac{\delta_2}{4\pi} N^2 (N+1)^2\ 2AN^{2\vartheta+1/2+2\varepsilon}.$$

## 12. The value of $v(t)$

According to (5.3) we had:

$$v_m(t-1) = L\text{-}1\left\{\left(\frac{1-e^{-s}}{s}\right)^2 \{T_1(s) + T_2(s) + T_3\}\right\} = F_1(t) + F_2(t) + A_1 N^{4\vartheta+1+2\varepsilon}$$

due to (10.11), (11.1) and (8.3).

Hence:

(12.1)
$$v_m(t-1) = P(t-0) \left\{\frac{t}{2} - \frac{1}{3}\right\} + P(t-1+0) \left\{\frac{t}{2} - \frac{2}{3}\right\} + \delta_4\, R(t-\delta_5) + \delta_6\, R(t-\delta_7) +$$

$$+\frac{\delta_2}{4\pi}N^2(N+1)^2\ 2AN^{2\vartheta+1/2+2\varepsilon}+A_1N^{4\vartheta+1+2\varepsilon}=$$

$$=P(t-0)\left\{\frac{t}{2}-\frac{1}{3}\right\}+P(t-1+0)\left\{\frac{t}{2}-\frac{2}{3}\right\}+$$

$$+\delta_4\delta_3e^{3\gamma}\ d(t-\delta_5)\frac{(\log\log N)^2}{N}\log\log(t-\delta_3)+\delta_6\delta_3e^{3\gamma}\ d(t-\delta_7)\frac{(\log\log N)^2}{N}\,.$$

$$\cdot\ \log\log(t-\delta_7)+\frac{\delta_2}{4\pi}N^2(N+1)^2\ AN^{2\vartheta+1/2+2\varepsilon}+A_1N^{4\vartheta+1+2\varepsilon}=$$

$$=\frac{\nu(t-1+0)+\nu(t-1-0)}{2}\,.$$

Replacing $t-1$ by $t$, and equating terms in $t\pm 0$ on both sides we deduce:

$$\nu(t)=P(t)\,(t+1)+2\delta_3\delta_4\,e^{3\gamma}\ d(t)\,\frac{(\log\log N)^2}{N}\log\log(t+1-\delta_5)\,- \tag{12.2}$$

$$-\frac{2}{3}P(t)+\frac{\delta_2}{2\pi}N^2(N+1)^2\ AN^{2\vartheta+1/2+2\varepsilon}+2A_1N^{4\vartheta+1+2\varepsilon}\,.$$

The terms in $P(t)$ only can be absorbed by changing slightly the values of $\delta_2$ and $\delta_3$, so we ignore them in what follows.

As in (12.2) we assume $P(t)\neq 0$. Then, according to (6.3) and (6.4) we must assume, in what follows, that $t$ is an even number.

## 13. Choosing N as a function of $t$

We choose now:

$$t=N^{5+2\vartheta+1/2}\qquad\qquad N=t^{\frac{1}{5+2\vartheta+1/2}} \tag{13.1}$$

so that

$$N+1<(t+1)^{\frac{1}{5+2\vartheta+1/2}}$$

Then

$$\nu(t)=P(t)\,t+\delta_8\,2e^{3\gamma}\,d(t)\,(\log\log t)^3\,t^{1-1/(5+2\vartheta+1/2)}+ \tag{13.2}$$

$$+\delta_2\frac{80}{2\pi}(t+1)^{\frac{4.5+2\delta+2\varepsilon}{5+2\vartheta+1/2}}-160t^{\frac{4\vartheta+1+2\varepsilon}{5+2\vartheta+\frac{1}{2}}}\,.$$

Hence :

$$\nu(t)>P(t)\,t-2e^{3\gamma}\,d(t)\,(\log\log t)^3\,t^{1-1/2\,(5+2\vartheta+1/2)}$$

$$- \frac{80}{2\pi}(t+1)^{1-\frac{1/2}{5+2\vartheta+1/2}} - 160\ t^{\frac{4\vartheta+1+2\varepsilon}{5+2\vartheta+1/2}}$$

Adopting $\vartheta = 1$ implies that

(13.3) $$\nu(t) > P(t)\,t - 2e^{3\gamma}\,d(t)\,(\log\log t)^3\ t^{1-\frac{1}{7.5}}$$

$$- \frac{80}{2\pi}(t+1)^{1-1/5} - 60t^{\frac{5+2\varepsilon}{7.5}}\ .$$

(13.2) proves unconditionally the truth of the Hardy – Littlewood conjectural formula for the binary Goldbach problem, that

$$\nu(t) \approx P(t)\cdot t$$

## 14. Miscellaneous bounds and majorizations

We have evidently, from (6.4)

(14.1) $$\min D(t) \geq 1.3203\ t\ .$$

As concerns $d(t)$ we have:

(14.2) $$\limsup_{t\to\infty} d(t) = d_0(t) = t^{\frac{\log 2}{\log\log t}}$$ (ref. [4]) .

In what follows we shall use very large values of $t$, so that we adopt the right hand side as an upper bound for $d(t)$.

Furthermore, the greatest value that $\nu(t)$ can assume in the case that there is only one solution $2\log^2 t/_2$ (as was indicated in ref. [1]).

Consequently, if we know that

(14.3) $$\nu\,(t) > 2\log^2 t/2$$

we can be sure that there is at least one solution.

Combining (13.2) and (14.3), we have that

(14.4) $$\nu(t) > 1.3203\ t - 2\,e^{3\gamma}\ d(t)\,(\log\log t)^3\ t^{1-1/7.5}$$

$$-\frac{80}{\pi}(t+1)^{1-1/15} - 160t^{\frac{5+2\varepsilon}{7.5}} > 2\log^2 t/2\,.$$

Solving for $d(t)$, we see that this holds if

(14.5) $$d\,(t) < \frac{1.3203}{2e^{\gamma}}\,\frac{t^{1/7.5}}{(\log\log t)^3} - \frac{80}{2\pi e^{3\gamma}}\,\frac{(t+1)^{1/15}}{(\log\log t)^3}$$

or if

(14.6) $$d(t) < 0.05846 \frac{t^{1/7.5}}{(\log \log t)^3}$$

which according to E.R.H. that $\vartheta = ½$, strengthens to

(14.7) $$d(t) < 0.05846 \frac{t^{1/6.5}}{(\log \log t)^3} .$$

Landau (ref. [5]) had proved that

$$d(t) < 4 \sqrt[3]{t}$$

From (14.6) and (14.2) it follows that the Goldbach hypothesis is valid for every value of $t$ such that

(14.8) $$d(t) \geq d_0(t) .$$

If we adopt the equality sign, we have an equation whose root is

(14.9) $$t = t_0 \sim 10^B \qquad 180 < B < 190 .$$

Hence the hypothesis is true for even $t > 10^B$. On the other hand, the conjecture has been verified numerically up to $t = t_1 = 4 \times 10^{11}$.

Besides, it is known (from Dirichlet's divisor problem), that the average value $\overline{d(t)}$ of $d(t)$ is:

(14.10) $$\overline{d(t)} = \log t .$$

If we put this value in (14.5), we then have an inequality that is valid for $t > t_2 \sim 10^{40}$. The distribution of the values of $d(t)$ around the average $\overline{d(t)}$ is not known at present, but at any rate it is evident that values as high as (14.2) are extremely rare, and that due to (14.7) the Goldbach hypothesis is valid for most even values of $t$ between $t_2$ and $t_0$.

Better approximations of the formulas involved will allow us, in the coming years, to imporve the proceding bounds and, eventually, prove the hypothesis for every even $t \neq 2$. By way of comparison, it can be mentioned that in the ternary Goldbach problem (that every odd number > 5 can be written as the sum of three primes) the existence of $N_0$, a number such that every odd number not 3 or 5 which is $> N_0$, is the sum of 3 primes, was proved only in 1937, by I.M. Vinogradov, and only in 1989 was it proved that $N_0 = \exp(\exp 11503)$ (ref.[8]).

**Improved Lemma 1**

According to ref. [9] we have:

(I.1) $$\sum_{q>N}^{\infty} \frac{\mu^2(q)}{\varphi^q(q)} C_q(t) = \sum_{d|t} \frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{N|d<q \\ (q,t)=1}} \frac{\mu(q)}{\varphi^2(q)} .$$

Now

(L2) $$\sum_{\substack{N|d<q \\ (q,t)=1}} \left|\frac{\mu(q)}{\varphi^2(q)}\right| < \sum_{\substack{N|d<q \\ (q,t)=1}} \frac{1}{\varphi^2(q)} < \sum_{N|d<q} \frac{1}{\varphi^2(q)}.$$

But for $\varphi(n)$ we have the bound

(L3) $$\frac{n}{\varphi(n)} \le e^{\gamma} \log\log n + \frac{5}{2\log\log n} \qquad \text{(ref. [8])}$$

valid for every $n \ge 3$ (with one exception), so that

$$\sum_{N|d<q} \frac{1}{\varphi^2(q)} < \sum_{N|d<q} e^{2\gamma} \frac{(\log\log n)^2}{n^2} + \frac{5}{2} \sum_{N|d<q} \frac{1}{n^2 \log\log n}$$

$$< e^{2\gamma} \int_{N|d-1}^{\infty} \frac{(\log\log u)^2}{u^2} du + \frac{5}{2} \int_{N|d-1}^{\infty} \frac{du}{u^2 \log\log u}$$

(L4) $$\approx e^{2\gamma} \frac{(\log\log \mathrm{N}/d)^2}{(\mathrm{N}/d)} + \frac{5}{2} \frac{1}{\log\log \mathrm{N}/d \cdot \mathrm{N}/d} .$$

Replacing (L4) in (L2) we obtain:

(L5) $$\sum_{\substack{N|d<q \\ (q,t)=1}} \left|\frac{\mu(q)}{\varphi^2(q)}\right| < e^{2\gamma} \frac{(\log\log \mathrm{N}/d)^2}{\mathrm{N}} \cdot d + \frac{5}{2} \frac{d}{\log\log \mathrm{N}/d \cdot \mathrm{N}} .$$

Replacing (L5) in (L1) we see that

$$\sum_{q>N}^{\infty} \left|\frac{\mu^2(q)}{\varphi^2(q)} C_q(t)\right| < \sum_{d|t} \frac{\mu^2(d)}{\varphi(d)} \left\{ e^{2\gamma} \frac{(\log\log \mathrm{N}/d)^2 d}{\mathrm{N}} + \frac{5}{2} \frac{d}{\mathrm{N} \log\log \mathrm{N}/d} \right\}$$

$$< \sum_{d/t} e^{2\gamma} \frac{(\log\log \mathrm{N}/d)^2}{\mathrm{N}} e^{\gamma} \left( \log\log d + \frac{5}{2\log\log d} \right)$$

$$< d(t)\, e^{3\gamma} \frac{(\log\log \mathrm{N})^2}{\mathrm{N}} \log\log t .$$

**Lemma 2.** Evaluation of $\Sigma\,\Sigma\,\Sigma\,\Sigma\, \dfrac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)} \cdot \dfrac{e^{-\mathrm{A}_1 t} - e^{-\mathrm{A}_2 t}}{\mathrm{A}_2 - \mathrm{A}_1}$

We have that

$$\frac{1}{\mathrm{A}_2 - \mathrm{A}_1} = \frac{1}{2\pi i} \frac{1}{\left(\frac{h_2}{q_2} - \frac{h_1}{q_1}\right)} = \frac{q_1 q_2}{2\pi i (h_2 q_1 - h_1 q_2)} .$$

Hence

$$\frac{1}{A_2 - A_1} = \frac{q_1\, q_2}{2\pi\, |h_2\, q_1 - h_1\, q_2|} \le \frac{q_1\, q_2}{2\pi}$$

because $|h_2\, q_1 - h_1\, q_2|$ is an integer whose least value is 1.

Furthermore

$$\left| e^{-A_1 t} - e^{-A_2 t} \right| \le 2$$

so that

$$\frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \le \frac{q_1\, q_2}{\pi}$$

and

$$\sum_{h_1=0}^{q_1-1} \sum_{h_2=0}^{q_2-1} \left| \frac{\mu(q_1)\, \mu(q_2)}{\varphi(q_1)\, \varphi(q_2)} \, \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| \le \frac{q_1\, q_2}{\pi}$$

because the double sum has $\varphi(q_1)\ \varphi(q_2)$ terms.

It follows that

$$\sum_{q_1\text{-}1}^{N} \sum_{h_1=0}^{q_1-1} \sum_{q_2\text{-}1}^{N} \sum_{h_2=0}^{q_2-1} \left| \frac{\mu(q_1)\, \mu(q_2)}{\varphi(q_1)\, \varphi(q_2)} \, \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| <$$

$$< \frac{1}{\pi} \sum_{q_1\text{-}1}^{N} \sum_{q_2\text{-}1}^{N} q_1\, q_2 = \frac{1}{\pi} \, \frac{N(N+1)}{2} \, \frac{N(N+1)}{2} = \frac{N^2\, (N+1)^2}{4\,\pi} .$$

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