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THE FUNDAMENTAL SOLUTION OF THE EQUATION $X^2 - DY^2 = 1$

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ABSTRACT. In this paper the authors use Analytic Number Theory tools to find the fundamental solution of the equation $X^2 - DY^2 = 1$.
Algebraicaly this equation is related with units in algebraic number quadratic fields.

1. Introduction

This Diophantine equation, wrongly known as "Pell's equation" since it is Euler's equation, appears for the first time in the history of Mathematics in Archimedes' (287 – 212 BC) cattle problem. For this reason let us call it Euler-Pellian equation.

In it the value of D turns out to be:

$$D = 2 \times 3 \times 7 \times 11 \times 29 \times 353 \times (2 \times 4657)^2 = 41.028.642.327.824$$

and the smallest solution has 206545 digits.

In later times Brahmagupta (born 598 AD) found the numerical solution $1151^2 - 92.120^2 = 1$, while Bhasca Acharya (born 1114 AD) deviced the "cyclic method" and the "composition method" to derive infinitely many solutions.

In 1657 Fermat, who was not aware of these ancient facts, made as a "second défi" to the English mathematicians to derive a method that would give infinitely many solutions.

Wallis and Lord Brouncker succeeded in finding a new method that tentatively solved the question.

Euler devoted many papers to the problem. He showed that wallis' method could be more convenient exhibited by means of the continued fraction for $\sqrt{D}$, because the successive reduced fractions of it are solutions of the equation.

Lagrange gave in 1766 the first proof that the equation has actually many solutions, if D is not a square, and proved the formula

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$$x_n + y_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n$$

that gives the relation among x_1, y_1 (the minimal solution) and x_n, y_n (the n-th solution), and showed that it gives all the possible solutions.

This equation has a pecial importance because the solution of the general equation of the second degree in two unknowns (when there are infinitely many solutions), depends upon it.

In 1806 Gauss published his most famous book "Disquisitiones Arithmeticorum", where the central problem is the solution of the second degree binary equation. His contributions were outstanding ones, but however he failed to find the size of the fundamental unit.

Hasse-Bernstein and Baica [1] obtained fundamental units related with the algorithmic solution of Euler-Pellian equation in the quadratic algebraic number fields.

(1.2) $$\eta_D = x_1 + \sqrt{D}\ y_1$$

in explicit asymptotic form.

In 1832 Jacobi stated that the solutions can be expressed in terms of the sine and cosine of $2m\pi/D$, while Dirichlet found the formula

(1.3) $$h(D) = \left(2\sqrt{D}/\log\eta_D\right)L_D(1)$$

where h(D) is the class number of the integral properly primitive binary quadratic forms $ax^2 + 2bxy + cy^2$ of determinant $D = b^2 - ac > 0$, and

(1.4) $$L_D(1) = \sum_{\substack{m=1 \\ m\ \text{odd}}} \left(\frac{D}{m}\right)/m$$

In 1863 Kronecker noted that $\log\eta_D$ can be expressed in terms of special theta functions or elliptic functions, and the number of classes of binary quadratic forms of determinant D.

At present, in absence of an analytical asymptotic formula for η_D, the interest is centred in the evaluation of the lengths of the algorithms that determine it (see ref.[6]).

2. A formula for $\Delta\left\{\left[\sqrt{t}\right]\right\}$

In his paper about cuboids, published in [7], A. Peretti proves the following formula:

For the function

(2.1) $$\Delta\{[\sqrt{t}]\} = [\sqrt{t}] - [\sqrt{t-1}] = \begin{cases} 1 & \text{if } n^2 = t < n^2+1 \\ & (n : \text{int egernumber}) \\ 0 & \text{in any other case} \end{cases}$$

we have the following asymptotic expansion:

(2.2) $$\Delta\{[\sqrt{t}]\} : \frac{t^{-1/2}}{2} \sum_{q=1}^{N} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \frac{G(q,h)}{q} e^{-2\pi i \frac{h}{q} t}$$

where

(2.3) $$G(q,h) = \sum_{x=0}^{q-1} e^{2\pi i x^2 \frac{h}{q}}$$

is a Gauss sum.

The evaluation of this sum is due to Dirichlet and Bachmann, and we have:

(2.4)
$$\begin{array}{ll}
\text{If } q = 2 & G(q,h) = 0 \\
\text{If } q = 2^{2\mu+1} & G(q,h) = 2^{\mu+1} e^{\frac{\pi}{4} h i} \\
\text{If } q = 2^{2\mu} & G(q,h) = 2^{\mu}(1+i^h) = 2^{\mu+1} \cos\frac{h\pi}{4}\, e^{i\frac{\pi}{4}h} \\
\text{If } q = p = 2\nu+1 & G(q,h) = \left(\frac{h}{p}\right) i^{\frac{(p-1)^2}{4}} \sqrt{p} = \left(\frac{h}{p}\right) e^{i\frac{\pi}{4}\nu} \sqrt{p} \\
\text{If } q = p^{2\mu+1} & G(q,h) = p^{\mu}\, G(p,h) \\
\text{If } q = p^{2\mu} & G(q,h) = p^{\mu}
\end{array}$$

with these values at hand, we can evaluate G(q,h) for any other composite q.

Remark an unnoticed property of these G(q,h): holds in every case that

(2.5) $$|G(q,h)| \le \sqrt{q}$$

except when $q = 2^{\mu+1}$, where $|G(q,h)| = \sqrt{2}\sqrt{q}$, and when $q = 2^{\mu}$, when $|G(q,h)| = 2\sqrt{q}$.

Next, in (2.2) we denote

(2.6) $$S_q(t) = \sum_{h=0}^{q-1} G(q,h)\, e^{-2\pi i \frac{h}{q} t}$$

with the introduction of this function in (2.2), it can be written as: .

(2.7) $$\Delta\{[\sqrt{t}]\} : \frac{t^{-1/2}}{2} \sum_{q=1}^{N} \frac{S_q(t)}{q}$$

The first terms of this formula are:

$$\Delta\left\{\left[\sqrt{t}\right]\right\}: \frac{t^{-1/2}}{2}\left\{1+O+\frac{1}{\sqrt{3}}\left\{\cos(\pi/4-2\pi t/3)-\cos(\pi/4-4\pi t/3)+O\right\}+\right.$$

$$+\frac{4}{\sqrt{4}}\left\{\cos^2\pi/4.\cos\pi t/2+\cos^2\pi/2.\cos\pi t+\cos^2 3\pi/4.\cos(3\pi t/2)+\cos^2\pi.\cos 2\pi t\right\}-$$

$$-\frac{1}{\sqrt{5}}\left\{\cos(2\pi t/5)-\cos(4\pi t/5)-\cos(6\pi t/5)+\cos(8\pi t/5)+O\right\}+....$$

3. A formula for N(Y).

If we denote with N(Y) the quantity of solutions of the equation

(3.1) $$x^2-Dy^2=1$$

with $y \le Y$,

then an obvious consequence of (2.1) is that

(3.2) $$N(Y)=\sum_{y\le Y}\Delta\left\{\left[\sqrt{Dy^2+1}\right]\right\}$$

In order to put N(Y) under the form of an integral, we can not use the Euler-Maclaurin sum formula (which would be at first sight the natural way), because the function at right does not posses a derivative.

Consequently, we use the Poisson summation formula, but not in its usual form, but in the following one (to be found in ref.[2]):

(3.3) $$\frac{1}{2}\sum_{r=A}^{r=B}{}'\left\{f(r+o)+f(r-o)\right\}=\int_A^B f(x).dx+2\sum_{m=1}^{\infty}\int_A^B f(x).\cos(2m\pi x)dx$$

where the dash in the summation sign indicates that for the terms corresponding to r =A and $r=B$, only $f(A+o)$ and $f(B+o)$, respectively, are counted.

The only restriction on f(r) is that it be of bounded variation in [A,B] we choose in the above formula

$$f(y)=\Delta\left\{\left[\sqrt{Dy^2+1}\right]\right\}$$

and we obtain from (2.5) that

(3.4) $$N(Y)=\int_1^Y\Delta\left\{\left[\sqrt{Dy^2+1}\right]\right\}dy\ +2\sum_{m=1}^{\infty}\int_1^Y\Delta\left\{\left[\sqrt{Dy^2+1}\right]\right\}\cos 2m\pi y.dy$$

(leaving aside an eventual term $\frac{1}{2}f(Y)$ that can appear in the count of (2.6))

4. Evaluation of $T_1=\int_1^Y\Delta\left\{\left[\sqrt{Dy^2+1}\right]\right\}.dy$

Replacement of t by Dy^2+1 in formula (2.2) yields:

(4.1) $$\Delta\left\{\left[\sqrt{Dy^2+1}\right]\right\}\sim\frac{\left(Dy^2+1\right)^{-1/2}}{2}\sum_{q=1}^{N}\frac{S_q\left(\sqrt{Dy^2+1}\right)}{q}$$

so that

$$T_1 \sim \frac{1}{2}\int_1^Y (Dy^2+1)^{-1/2}\sum_{q=1}^{N}\frac{S_q\left(\sqrt{Dy^2+1}\right)}{q}\,dy$$

(4.2)
$$\sim \frac{1}{4\sqrt{D}}\int_1^{DY^2}\frac{1}{\sqrt{u(u+1)}}\sum_{q=1}^{N}\frac{S_q\left(\sqrt{u+1}\right)}{q}\,du$$

$$\sim \frac{1}{4\sqrt{D}}\int_1^{DY^2}\sum_{q=1}^{N}\frac{S_q\left(\sqrt{u+1}\right)}{q}\,\frac{du}{u}$$

Now, the first mean value theorem of the integral calculus states that if f(x) is a continuous function, g(x) is integrable, $m \le f(x) \le M$, and g(x) does not change sign in the interval (A,B), then there exist at least a value ξ $(A \le \xi \le B)$ such that

$$\int_A^B f(x)\,g(x)\,dx = f(\xi)\int_A^B g(x)\,dx$$

We apply this theorem to the integral in (3.2) choosing $g(x) = 1/x$ and we obtain:

(4.3)
$$T_1 = \frac{1}{4\sqrt{D}}\,S(\xi)\,\log DY^2 \qquad (\text{with } 1 \le \xi \le DY^2)$$

where

(4.4)
$$S(\xi) = \sum_{q=1}^{N}\frac{S_q\left(\sqrt{\xi^2+1}\right)}{q}$$

is the singular series of the problem.

5. Evaluation of $T_2 = 4\sum_{m=1}^{\infty}\int_1^Y \Delta\left\{\left[\sqrt{Dy^2+1}\right]\right\}.\cos 2m\pi y.dy$

According to (2.1) we have:

(5.1)
$$\int_1^Y \Delta\left\{\left[\sqrt{Dy^2+1}\right]\right\}.\cos 2m\pi y.dy = \sum_{n=1}^{Y}\int_{n^2}^{n^2+1} 1.\cos 2m\pi y.dy =$$

$$= \sum_{n=1}^{Y}\frac{\sin 2m\pi(n^2+1)}{2m\pi} - \sum_{n=1}^{Y}\frac{\sin 2m\pi n^2}{2m\pi} = 0$$

(5.2) so that $T_2 = 0$

6. Final result

Collecting the partial results (2.5), (2.7), (3.3) and (4.2) we deduce:

(6.1) $$N(Y) \sim \frac{S(\xi)}{4\sqrt{D}} \log DY^2$$

7. The asymptotic values of the solutions, for large Y and D.

They can be deduced from formula (5.1). Very clearly $N(y_n) = n$, so that we obtain from (5.1) that

(7.1) $$n \sim \frac{S(\xi)}{4\sqrt{D}} \log\left(D y_n^2\right)$$

Passing to exponentials we get

(7.2) $$\sqrt{D}\, y_n \sim \exp\left\{\frac{2n\sqrt{D}}{S(\xi)}\right\}$$

and

(7.3) $$x_n = \sqrt{Dy_n^2 + 1} \sim \left\{\exp \frac{4n\sqrt{D}}{S(\xi)} + 1\right\}^{1/2} \sim \exp \frac{2n\sqrt{D}}{S(\xi)}$$

The minimal solution, for large D, is

(7.4) $$y_1 \sim \frac{1}{\sqrt{D}} \exp\left\{\frac{2\sqrt{D}}{S(\xi)}\right\} \qquad x_1 \sim \exp\left\{\frac{2\sqrt{D}}{S(\xi)}\right\}$$

so that the fundamental solution

$$\eta_D = x_1 + y_1\sqrt{D}$$

is asymptotically equal to

(7.5) $$\eta_D \sim 2 \exp\left\{\frac{2\sqrt{D}}{S(\xi)}\right\}$$

Due to the oscillating value of $S(\xi)$, formula (6.2) explains the violent changes that are found in the values of table 1.

From (6.5) follows that

(7.6) $$\log \eta_D \sim \frac{2\sqrt{D}}{S(\xi)} + \log 2$$

8. Bounds for $S(\xi)$

At present, we have not any reasonable way to evaluate $S(\xi)$, nor η_D.

However, a certain orientation can be derived from following considerations.

The equation $x^2 - \left(k^2 - 1\right)y^2 = 1$ has the minimal solution $x_1 = k \quad y_1 = 1$, so that

(8.1) $$\eta_D = k + \sqrt{k^2 - 1} \sim 2k \sim 2\sqrt{D}$$

But from (6.6) follows that

(8.2) $$\eta_D \sim 2\exp\left(\frac{2\sqrt{D}}{S(\xi)}\right)$$

Equating (7.1) and (7.2) we obtain:

(8.3) $$S(\xi) \sim \frac{2\sqrt{D}}{\log\sqrt{D}} \sim 2D^{1/2-\varepsilon}$$

Besides Hooley has proved the following theorem:

- Denote with $\Gamma(x,\alpha)$ the quantity of $D \le x$ such that $\eta_D \le D^{1/2+\alpha}$ with $0 \le \alpha \le 1/2$. Then $\Gamma(x,\alpha) \sim \frac{4\alpha^2}{\pi^2}\sqrt{x}\,.\log^2 x$
- There is a positive density of determinants D for which $\eta_D > D^{3/2} / \log D$ (ref.[4])
- If $b < 3/4$, then $\eta_D > D^b$ for almost every D

On the other side L. K. Hua (ref.[5]) proved that for the equation

$$x^2 - Dy^2 = 4$$

holds the bound

$$\eta_D < 2\exp\left\{\sqrt{D}\left(\frac{D}{2}+1\right)\right\}$$

It can be conjectured that this bound is also of application to Dell's equation. However this bound is not better than the older one obtained by Thekla Schmitz (ref.[7]), that

$\eta_D < 2\exp(4D)$

Table 1. Smallest solutions y_1 of $Dy^2 + 1 = \text{square}$

D	y_1	D	y_1	D	y_1
1	-	51	7	101	20
2	2	52	90	102	10
3	1	53	9100	103	22419
4	-	54	66	104	5
5	4	55	12	105	4
6	2	56	2	106	3115890
7	3	57	20	107	93
8	1	58	2574	108	130
9	-	59	69	109	15140424455100
10	6	60	4	110	2
11	3	61	226153980	111	28
12	2	62	8	112	12
13	180	63	1	113	113296
14	4	64	-	114	96
15	1	65	16	115	105
16	-	66	8	116	910
17	8	67	5964	117	60
18	4	68	4	118	28254
19	39	69	936	119	11
20	2	70	30	120	1
21	12	71	413	121	-
22	42	72	2	122	22
23	5	73	267000	123	11
24	1	74	430	124	414960
25	-	75	3	125	83204
26	10	76	6630	126	40
27	5	77	40	127	419775
28	24	78	6	128	51
29	1820	79	9	129	1484
30	2	80	1	130	570
31	273	81	-	131	927
32	2	82	18	132	2
33	4	83	9	133	224460
34	6	84	6	134	12606
35	1	85	30996	135	21
36	-	86	1122	136	3
37	12	87	3	137	519712
38	6	88	21	138	4
39	4	89	53000	139	6578829
40	3	90	2	140	6
41	220	91	165	141	8
42	2	92	120	142	12
43	531	93	1260	143	1
44	30	94	221064	144	-
45	24	95	4	145	24
46	3588	96	5	146	12
47	7	97	6377352	147	8
48	1	98	10	148	6
49	-	99	1	149	2113761020
50	14	100	-	150	4

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