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HERMITE'S PROBLEM FROM THE PERIODICITY OF (ACF) ALGORITHM

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ABSTRACT

There is a one-to-one correspondence between irrationals $\alpha > 1$ and infinite sequences (continued fractions) of the from $\{a_0, a_1, ..., a_n, ...\}$

Euler-Lagrange proved that the sequence $\{a_0, a_1, ..., a_n, ...\}$ is periodic if and only if α is quadratic irrational (is algebraic of second degree).

The periodicity of this sequence follows from the periodicity of the Euclidean Algorithm.

Hermite asked if there is a periodic algorithm to develop irrationals of any degree into such periodic sequences $\{a_0, a_1, ..., a_n, ...\}$.

Jacobi, Perron, Hasse and Bernstein worked on this problem. From the periodicity of Jacobi-Perron Algorithm or (abbr. JPA) and its modifications they solved Hermite's Problem for some higher degree irrationals.

In this paper the author will use the periodicity of ACF algorithm, previously described in another paper, to extend Hermite's Problem over the complex numbers. From the periodicity of ACF additional higher degree irrationals will be shown to have a periodic sequence development.

This ACF will not completely solve Hermite's Problem, wich may remain an open question for years to come, but these results are one more step in this direction.

0. INTRODUCTION

The objective of this paper is the development of an irrational number of any degree into a periodic sequence. This problem of Hermite has been open since the middle of the nineteenth century.

The beginning of this subject is the well known Euclidean Algorithm (EA). For example, by using this algorithm it is easy to prove that every rational number, a/b, can be represented as a finite simple continued fraction. In 1737 Euler proved that every real quadratic irrational can be represented by an infinite periodic continued fraction or by a periodic EA development. The converse was proved by Lagrange in 1770.

Of course if the number is not a quadratic irrational, but is a real algebraic number of higher degree or a transcendental real number, then its development by the EA cannot be periodic.

In 1839, Hermite, in one of his letters to Jacobi, challenged Jacobi to find an algorithm to develop irrationals of any degree into periodic sequences. But it was only after thirty years of frustration that Jacobi in 1869 extended EA methods to successfully represent some cubic irrationals by means of simple continued fractions. Then if 1907, Perron generalized the work of Jacobi. This generalization is known as the Jacobi - Perron Algorithm (JPA) and with it, Jacobi exhibited periodic developments for the cube roots of the numbers two, three and five. Advances were slow and difficult, but in 1873 Bachman proved results for other cubic irrationals using the JPA results that were accompanied by many restrictions. It is odd that no more progress occurred on these problems until Hasse and Bernstein turned their attention to them in their paper [6].

In its general form, as defined by Jacobi, an application of the JPA starts with the definition of an initial vector, the components of which are algebraic numbers. By use of the greatest integer function a "companion vector" is defined. A recursive transformation is constructed and applied to these vectors. For good choices of the starting vector and for the transformation, the iteration of the transformation becomes periodic, that is the transformation cycles around a finite set of vectors. In this instance the JPA is said to be periodic, and the results lead to the periodic representation of higher degree irrationals.

The difficulties associated with this work are many. Jacobi's results were confined to a few numerical examples in a cubic field. Perron generalized the method to apply to irrationals of any degree, but since the choices of starting vector and transformation are difficult to make, he was also limited to a few periodic developments of higher degree irrationals. Perron was more successful in showing that if a development is periodic, then the components of the initial vector are algebraic numbers. This latter result was general.

With Perron's work in 1907 on Hermite's Problem progress came to a halt, because of the failure of the JPA to produce new numerical results, that is additional cases in which the transformation becomes periodic were not achieved. Perron and all others recognized that the usual choices for starting vector were too limited.

No results were accomplished between 1907 and 1965, when Hasse and Bernstein made a broader approach to the periodicity problem associated with the JPA. Hasse and Bernstein were not interested in Hermite's problem and confined their work to searching for units in algebraic number fields.

Hasse and Bernstein started with an algebraic extension of the rational numbers, Q(w), where w takes the form $w = \sqrt[n]{D^n + d}$ with

$$P(x) = \left(\prod_{i=1}^{n} (x^n - D_i^n)\right) - d,\ d \varepsilon Z,\ D_i \varepsilon N \text{ and } d \mid D_i.$$

$$a^{(0)} = ((w - D_1)(w - D_2) \dots (w - D_{n-1}), \dots, (w - D_1)(w - D_2), (w - D_2))$$

$$\text{with } b^{(0)} = a^{(0)}(D_1).$$

Hasse and Bernstein showed that certain restrictions on D and d led to a JPA that was purely periodic. For $d > 0$ they proved that the JPA of $a^{(0)}$ is purely periodic when $D \geq (n - 2)\, d$ and $n \geq 3$. For $d < 0$ the sequence is also purely periodic when $D \geq 2\,(n - 1)\, d$ and $n \geq 3$. The length of the period is $n(n - 1)$. For this approach the periodicity remains an open problem since there are bounds on D and the restriction d/D must hold. For example no periodicity for $w = \sqrt[5]{12^5 + 6}$ can be proved under the Hasse-Bernstein restrictions. It will be seen later that some of these conditions can be removed.

When the algorithm becomes periodic an algebraic field unit is generated by the product of the last components of all of the vectors in the cycle. This was a strong result in the theory of units in algebraic number fields. The number of fields in which the Hasse-Bernstein units can be found is tied to the restrictions on their application of the JPA. The Hasse-Bernstein results were limited by their choices of w as real numbers. It should be noted that Hasse and Bernstein were not interested in Hermite's problem even if their results had a strong relation to that problem. Specifically, they did not know that the periodicity of the algorithm leads to a solution of Hermite's Problem for some real algebraic number w.

The author [1] defined a modification of the JPA that used the Hasse-Bernstein initial vector, but was not restricted to the real numbers. For the first time the complex numbers were considered. The only differences in the definitions stated above are that the D_i – s are now complex. An immediate consequence of this extension is that the bounds on D in the Hasse-Bernstein work are now eliminated and only the divisibility condition, d|D, remains. Returning to the example cited above, it can now be seen that $w = \sqrt[5]{12^5 + 6}$ has a periodic development, only 6|12 is required.

The author named the extension, the Algorithm for Complex Numbers (ACF). All of the previous results of the JPA and all of the Hasse-Bernstein results are consequences of this modification. All units in algebraic number fields so far discovered are particular cases of the application of Baica's Algorithm for Complex Numbers. The fact that the new algorithm unifies all previous work in the theory of units makes it a partial solution to the problem of Hilbert that asks for a common periodic algorithm from which all units in all algebraic number fields can be derived.

In addition the new algorithm assists in the solution of higher degree Diophantine equations, provides surprising derivations of complicated combinatorial identities, generates n-dimensional Fibonacci numbers, as well as makes possible the algorithmic approximation of irrationals [2, 3, 4, 5].

Finally, ACF algorithm provides new progress toward the solution of Hermite's Problem. A closer approach to the total solution of Hermite's Problem will be made in which the only remaining barrier is the divisibility condition, d|D. In addition Hermite's Problem will be extended and restated to include the complex numbers. For the future this new algorithm (ACF) will be named the Generalized Euclidean algorithm (GEA).

The problem which we investigate here is to obtain a periodic GEA development for more higher degree irrationals.

1. THE GEA

JPA is very difficult to prove periodic. Many prominent mathematicians in the modern mathematical world gave various modifications of JPA in order to prove its periodicity, but the problem still remained an open question. The author [1] used a modification which holds for complex fields of any degree (ACF). In the sequel, it will be called GEA. With this GEA for the first time complex numbers not only real number were found to have a periodic GEA development. The GEA was described in [1] where the author proved its periodicity and some other important results. For now let us proceed as follows.

DEFINITION 1. Let

$$a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \dots, a_{n-1}^{(0)}) \in R^{n-1}, n \geq 1 \tag{1.1}$$

be fixed, given vector, and

$$< a^{(v)} >, \ v = 0,1, \dots, a^{(v)} \in R^{n-1}$$

a sequence of vectors in R^{n-1} either given by some rule or calculated from $a^{(0)}$,

$$a^{(v)} = (a_1^{(v)}, a_2^{(v)}, \dots, a_{n-1}^{(v)})$$

Let $\langle b^{(v)} \rangle$, $v = 0,1, \ldots$; $b^{(v)} \in R^{n-1}$ be another sequence of vectors in R^{n-1}, either given by some rule or calculated from $\langle a^{(v)} \rangle$ with the formula

$$(1.2) \qquad \begin{cases} a^{(v+1)} = \left(a_1^{(v)}, b_1^{(v)}\right)^{-1} \left(a_2^{(v)} - b_2^{(v)}, \ldots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad v = 0, 1, \ldots \end{cases}$$

We say that the GEA holds for $a^{(0)}$. $b^{(v)}$ is called the "companion vector" of $a^{(v)}$.

DEFINITION 2. Let the GEA of an $a^{(0)}$ hold. If there exist two numbers, $s \geq 0$, $t \geq 1$: $s, t \in N$ such that $a^{(s+t)} = a^{(t)}$, then the GEA of this $a^{(0)}$ is called "periodic". If contemporarily min $s = m \geq 0$, min $t = l \geq 1$, then $\langle a^{(v)} \rangle$, $v = 0,1, \ldots, m-1$, $\langle a^{(v)} \rangle$, $v = m, m+1, \ldots, m+l-1$ are called respectively the "primitive preperiod" and the "primitive period" of the GEA of $a^{(0)}$. If $m = 0$, the GEA of $a^{(0)}$ is called "purely periodic". m and l are called respectively the lengths of the primitive preperiod and primitive period.

The reader should note that if the GEA of some $a^{(0)}$ is periodic then there exists an $a^{(v)}$ in this GEA such that the GEA of $a^{(v)}$ is purely periodic. With this in mind the author [1] has proved.

THEOREM 1. Let

(1.3) w and n-th degree irrational ($n \geq 2$), and $a^{(0)}$ a fixed vector such that

$$(1.4) \qquad \begin{cases} a^{(0)} = \left(a_1^{(0)}(w), a_2^{(0)}(w), \ldots, a_{n-1}^{(0)}(w)\right) \\ a_i^{(0)}(w) \quad \text{algebraic integers } (i = 1, \ldots, n-1) \end{cases}$$

Let the GEA of $a^{(0)}$ be purely periodic with length of the primitive period $= l$ and let the components of the companion vectors be algebraic integers. Then

$$(1.5) \qquad \begin{cases} A_0^{(v\,l)} + a_1^{(0)} A_0^{(v\,l+1)} + a_2^{(0)} A_0^{(v\,l+2)} + \ldots + a_{n-1}^{(0)} A_0^{(v\,l+n-1)} \\ \qquad\qquad\qquad\qquad v = 1, 2, \ldots \end{cases}$$

are units, namely

$$A_0^{(l)} + a_1^{(0)} A_0^{(l+1)} + a_2^{(0)} A_0^{(l+2)} + \ldots + a_{n-1}^{(0)} A_0^{(l+n-1)}$$

In this context we need the formula used by the author in [1], viz.

$$(1.6) \qquad \prod_{i=1}^{k} a_{n-1}^{(i)} = A_0^{(k)} + a_1^{(k)} A_0^{(k+1)} + \ldots + a_{n-1}^{(k)} A_0^{(k+n-1)} .$$

In the GEA of $a^{(0)}$ in E_{n-1} is purely periodic with length of the primitive period l then it follows from (1.6)

$$(1.7) \qquad e = \prod_{i=0}^{l-1} a_{n-1}^{(i)} = \sum_{j=0}^{n-1} a_j^{(0)} A_0^{(l+j)} \text{ , e a unit}$$

since in this case $a_{n-1}^{(l)} = a_{n-1}^{(0)}$ in virtue of (1.5) we have

$$(1.8) \qquad \begin{cases} e^f = A_0^{(fl)} + a_1^{(0)} A_0^{(fl+1)} + \ldots + a_{n-1}^{(0)} A_0^{(fl+n-1)} \\ \qquad f = 1, 2, \ldots \end{cases}$$

2. GEA - PERIODIC DEVELOPMENT OF IRRATIONALS - CASE n = 2

Though this case is well known from the expansion of real quadratic irrationals as simple continued fractions, we shall include it in our discussion.

Let

$$(2.1) \qquad w = \sqrt{D^2 + 1} \text{ , } D \in N, \qquad w \text{ a quadratic irrational.}$$

That w is irrational (for $D > 0$) is trivial. We choose the fixed vector.

$$(2.2) \qquad a^{(0)} = w + D,$$

since here $n - 1 = 1$. Thus $a_1^{(0)} = a_{n-1}^{(0)}$ and we shall generally denote

$$(2.3) \qquad a^{(v)} = a_v, \ v = 0,1, \ldots; \ a_v = a_v(\alpha) \text{ for all GEA of } a^{(0)}.$$

By (2.3) we denote

$$b^{(v)} = b_v, \quad v = 0,1, \ldots ;$$

For the calculation of the companion vectors we use the rule

$$(2.4) \qquad b^{(v)} = b_v = a_v(D), \quad b = 0,1, \ldots$$

and have

(2.5) $b_0 = (w + D)_{w=D} = 2D.$

hence, by (2.1)

(2.6) $a_1 = [(w + D) - 2D]^{-1} \cdot 1 = (w - D)^{-1} = w + D$

since $(w - D)^{-1} = (w + D)$ from $(w^2 - D^2) = 1$. Thus

(2.7) $a_0 = a_1 = \dots = a_v, \quad v = 0,1, \dots$

and the GEA of $a_0 = w + D$ is purely periodic with length of the primitive period $l = 1$. Further.

(2.8) $[a_v] = [w + D] = [w] + D = 2D = b_v$

the GEA of $w + D$ coincides with the EA and we have, in the notation of continued fractions

(2.9) $$a_0 = w + D = [\,\overline{2D}\,]$$

(2.9) is the periodic GEA development of a quadratic irrational $a_0 = w + D$.

3. THE CASE n = 3

We denote

(3.1) $$w = \sqrt[3]{D^3 + 1}$$

and choose the fixed vector

(3.2) $$a^{(0)} = (w + 2D, w^2 + Dw + D^2)$$

with $a^{(0)} = (a_1^{(0)}(w), a_2^{(0)}(w))$.

We apply the rule for calculating the components of the companion vectors

$$b_i^{(v)} = a_i^{(v)}(D), \; i = 1, 2; \; v = 0, 1, \dots .$$

We proceed with the GEA of $a^{(0)}$

$$b^{(0)} = (D + 2D,\ D^2 + D.\ D + D^2)$$

(3.3) $$b^{(0)} = (3D,\ 3D^2).$$

$$a^{(1)} = (w + 2D - 3D)^{-1}\ (w^2 + Dw + D^2 - 3D^2,\ 1) =$$

$$= (2 - D)^{-1}\ (w^2 + Dw - 2D^2,\ 1) =$$

$$= (w - D)^{-1}\ ((w - D)\ (w + 2D),\ 1).$$

(3.4) $$a^{(1)} = (w + 2D,\ w^2 + Dw + D^2) = a^{(0)}.$$

By (3.4) the GEA of

$$a^{(0)} = ((w + 2D),\ w^2 + wD + D^2),\ w = \sqrt[3]{D^3 + 1}$$

is purely periodic and the length of its primitive period $l = 1$. Using the notation in (2.9) we have

(3.5) $$a_0 = (w + 2D,\ w^2 + Dw + D^2) = \left[\overline{3D},\ \overline{3D^2}\right]$$

This can be considered the development of the components of a_0 using the periodicity of GEA and those components are algebraic irrationals of third degree. This is not true for all third degree irrationals, but for all those third degree irrationals which are components of a starting vector $a^{(0)}$ that leads to a periodic GEA.

4. THE CASE n = 5

We denote again

(4.1) $$w = \sqrt[5]{D^5 + 1}$$

and choise the fixed vector

(4.2) $$a^{(0)} = (w + 4D,\ w^2 + 3wD + 6D^2,\ w^3 + 2w^2D + 3wD^2 + 4D^3,\ w^4 + Dw^3 + D^2w^2 + Dw^3 + D^4).$$

Using the previous argument we find

$$b^{(0)} = (5D, 10D^2, 10D^3, 5D^4) \text{ or}$$

(4.3)

$$b^{(0)} = \left(\binom{5}{1}D, \binom{5}{2}D^2, \binom{5}{3}D^3, \binom{5}{4}D^4\right).$$

Thus the GEA of $a^{(0)}$ is purely periodic with legths of primitive period $l = 1$ and like in (3.5)

(4.4) $$a_0 = \left[\overline{5D}, \overline{10D^2}, \overline{10D^3}, \overline{5D^4}\right].$$

This solves the problem for some fifth degree irrationals.

5. THE GENERAL CASE

Let w be the irrational

(5.1) $$w = \sqrt[n]{D^n + 1}\ ;\ n \geq 2,\ D \varepsilon N;$$

and choose the fixed vector

(5.2) $$\begin{cases} a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_s^{(0)}, \ldots, a_{n-1}^{(0)}) \\ a_s^{(0)} = \sum\limits_{i-0}^{s} \binom{n-s-1+i}{i} w^{s-i} D^i \\ s = 1, \ldots, n-1 \end{cases}$$

The proof that the GEA of the fixed vector $a^{(0)}$ like in (5.2) is purely periodic and the length of its primitive period is $l = 1$ was given by the author in [2]. Thus we find

(5.3) $$b^{(0)} = \left(\binom{n}{1}D, \binom{n}{2}D^2, \ldots, \binom{n}{n-1}D^{n-1}\right)$$

With (5.3) we denote

(5.4) $$a_0 = \left[\overline{nD}, \overline{\frac{n(n-1)}{2}D^2}, \ldots, \overline{nD^{n-1}}\right].$$

(5.4) will solve the problem for some n-th degree irrationals.

Since GEA was not proved periodic for any degree irrational w we cannot get a periodic lGEA for any degree irrationals. This GEA does not completely solve Hermite's Problem. which may remain an open question for years to come, but these results are one more step in this direction.

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