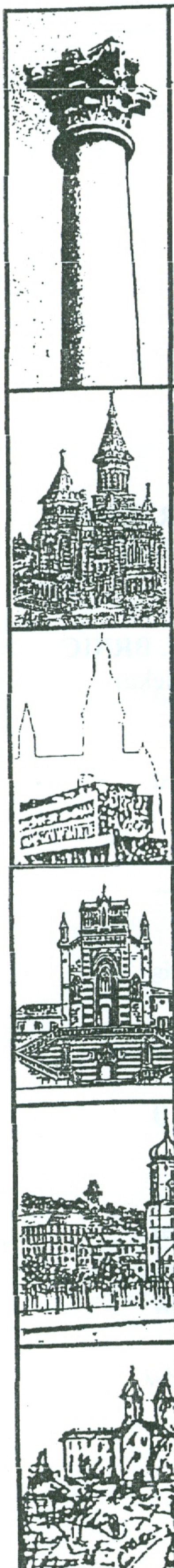

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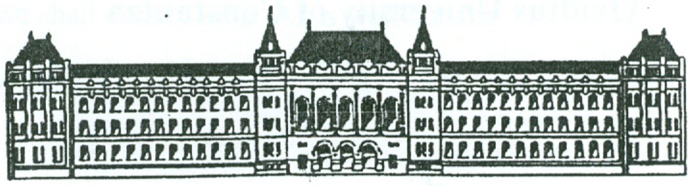

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SOME TRIGONOMETRIC IDENTITIES

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ABSTRACT

In a previous paper [1] the authour obtained new trigonometric identities of the form

$$2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2}\left(1-\cos\frac{2\pi k}{p}\right)^{p-1-k} = p^{p-2} \quad (1)$$

Which were derived as a result of relations in a cyclotomic field $R(\rho)$, where R is the field of rationals and ρ is a root of unity. The identities hold for every positive inteaer $p \geq 3$.

In this paper two formulas $\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\binom{p}{2k}\tan^{p-1-2k}\Phi = 0$ (2) and

$$-1+\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\left(\sum_{i=0}^{\frac{p-1-2k}{2}}\binom{p}{2k+2i}\binom{k+1}{k}\right)\cos^{p-2k}\Phi = 0 \quad (3)$$

are obtained and used in order to give some numeric applications of our new trigonometric identities (1).

Key words and phrases. Trigonometric identities, cyclotomic field.
1980 mathematics subtect classification code 12 A 35, 12 E 12.

0. INTRODUCTION

The trigonometric identities which were obtained in [1] are a result of relations in a cyclotomic field $R(\rho)$, where R is the field of rationals and ρ is a root of unity. The reader will be familiar with the primitive p-th roots of unity which are the p-1 different roots of the irreducible polynomial

$$x^{p-1} + x^{p-2} + \ldots + x + 1 = 0, \qquad p \text{ a prime} > 2 \qquad (0.1)$$

which we shall denote by ρ. As is well known

$$\rho = \cos\phi + i\sin\phi, \quad \phi = \frac{2\pi k}{p}, \qquad k = 1, 2, \ldots, k-1 \qquad (0.2)$$

Since the p −1 entities $\rho, \rho^2, ..., \rho^{p-1}$ form all the different roots of (0.1) we shall choose

$$\rho = \cos\phi + i\sin\phi, \quad \phi = \frac{2\pi}{p}, \quad (k=1) \tag{0.3}$$

The cyclotomic fields have been substantially investigated and the author will make use of some comments.

1. NUMERIC APPLICATION OF (1)

We shall give a few examples for the identity (1). For this purpose, we need a short review of known formulas.
Let

$$\sqrt[p]{1} = \cos\frac{2\pi k}{p} + i\sin\frac{2\pi k}{p}, k = 0,1,...,p-1$$

$$\sqrt[p]{1} = \left(\cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}\right)^k, k = 0,1,...,p-1$$

In (1), choosing k=1, $\frac{2\pi}{p} = \Phi$, we have

$$(\cos\phi + i\sin\phi)^p - 1 = 0$$

$$\cos^p\phi(i\tan\phi + 1)^p - 1 = 0$$

$$(i\tan\phi + 1)^p = \frac{1}{\cos\phi} = 0, \tag{1.1}$$

and for the imaginary part of (1.1) we obtain, after canceling by i;

$$\tan^p\phi = \binom{p}{2}\tan^{p-2}\phi + \binom{p}{4}\tan^{p-4}\phi + ... + (-1)^{\frac{p-1}{2}} p\tan\phi = 0 \tag{1.2}$$

Excluding $\tan\phi = 0$, we can dived (1.2) by $\tan\phi$ and obtain the equation for $\tan\phi$

$$\sum_{k=1}^{\frac{p-1}{2}} (-1)^k \binom{p}{2k} \tan^{p-1-2k}\Phi = 0 \tag{1.3}$$

This formula (2) was also stated by Gauss [2]. We shall still show how the $\cos\phi$ can be obtained from (1.3) since

$$\tan^2\alpha = \frac{1-\cos^2\alpha}{\cos^2\alpha}.$$

We obtain from (1.3)

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{p}{2k} \frac{(1-\cos^2\phi)^{\frac{p-1-2k}{2}}}{(\cos^2\phi)^{\frac{p-1-2k}{2}}} = 0 \tag{1.4}$$

Setting

$$\cos^2\phi = x \tag{1.5}$$

we obtain from (1.4)

$$\sum_{k-0}^{p-1}(-1)^k\binom{p}{2k}\frac{(1-x)^{\frac{p-1-2k}{2}}}{x^{\frac{p-1-2k}{2}}}=0 \tag{1.6}$$

and in expanded form

$$\frac{(1-x)^{\frac{p-1}{2}}}{x^{\frac{p-1}{2}}}-\binom{p}{2}\frac{(1-x)^{\frac{p-3}{2}}}{x^{\frac{p-3}{2}}}+\binom{p}{4}\frac{(1-x)^{\frac{p-5}{2}}}{x^{\frac{p-5}{2}}}-\ldots+(-1)^{\frac{p-1}{2}}p=0$$

or

$$\sum_{k=0}^{p-1}(-1)^k(1-x)^{\frac{p-1-2k}{2}}x^k\binom{p}{2k}=0. \tag{1.7}$$

This equation is of degree $\frac{p-1}{2}$ for $\cos^2\phi$. But there is another formula for $\cos\phi$ which is obtained in the following way. We obtain for the real part of

$$(\cos\phi + i\sin\phi)^8 - 1 = 0$$

$$-1+\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{p}{2k}\cos^{p-2k}\phi\sin^{2k}\phi=0$$

or

$$-1+\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{p}{2k}\cos^{p-2k}\phi(1-\cos^2\phi)^k=0$$

which is easily transformed into

$$-1+\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\left(\sum_{i-0}^{\frac{p-1-k}{2}}\binom{p}{2k+2i}\binom{k+1}{k}\right)\cos^{p-2k}\phi=0 \tag{1.9}$$

We obtain for the first element under the sigma sign with k=0

$$\sum_{i=0}^{\frac{p-1}{2}}\binom{p}{2i}=1+\binom{p}{2}+\binom{p}{4}+\ldots+\binom{p}{p-1}=2^{p-1}$$

Equation (1.9) is of degree p; one of its roots is, from $\frac{2\pi}{k}=\phi=0$ for k=0, cos0=1. so polynomial (1.9) is divisible by $\cos\phi-1$. The quatient polynomial is of degree p-1, its p-1 roots $\frac{2\pi}{p},\frac{2\cdot 2\pi}{p},\ldots,\frac{(p-1)2\pi}{p}$ are paiwise equal, vis.

$\cos\frac{2\pi k}{p}=\cos\left(2\pi-\frac{2\pi k}{p}\right), k=1,\ldots,p-1,$ hence this quotient polynomial is the perfect square of a polynomial of degree $\frac{p-1}{2}$, the roots of wjich are $\cos k\phi, k=1,\ldots,\frac{p-1}{2}$.

Formula (1.9) is also stated by Gauss [2] in a slightly different form. Another way to obtain $\cos\phi$ directly is obtained in the following method investigated also by Perron [3] for symmetric polynomial. We have the irreductible equation (0.1)

$$x^{p-1}+x^{p-2}+\ldots+x+1=0$$

Dividing by $x^{\frac{p-1}{2}}$ we obtain

$$\left(x^{\frac{p-1}{2}}+\frac{1}{x^{\frac{p-1}{2}}}\right)+\left(x^{\frac{p-3}{2}}+\frac{1}{x^{\frac{p-3}{2}}}\right)+\ldots+\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0 \qquad (1.10)$$

Setting $x+\frac{1}{x}=y$ we obtain easily

$$\left.\begin{aligned} x^2+\frac{1}{x^2}&=y^2-2,\\ x^3+\frac{1}{x^3}&=y^3-3y,\\ &\ldots\ldots\ldots\ldots\\ x^{\frac{p-1}{2}}+\frac{1}{x^{\frac{p-1}{2}}}&=y^{\frac{p-1}{2}}+\ldots \end{aligned}\right\} \qquad (1.11)$$

But $x+\frac{1}{x}=\cos\phi+i\sin\phi+\cos\phi-i\sin\phi=2\cos\phi=y,$ so that a root of (1.10) is $2\cos\phi$.

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