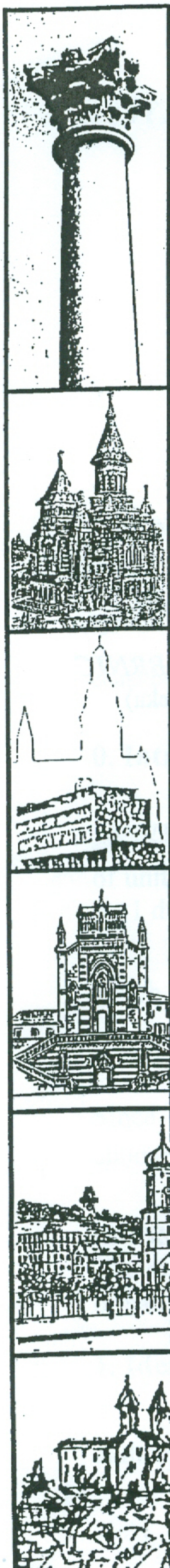

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A CYCLOTOMIC FIELD TO DERIVE SOME TRIGONOMETRIC IDENTITIES

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Written by

Malvina Baica

University of Winsonsin

Caretaken by the PAMM-Centre

at the

Technical University of Budapest

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A CYCLOTOMIC FIELD TO DERIVE SOME TRIGONOMETRIC IDENTITIES

by Malvina BAICA

ABSTRACT. In this paper the author obtains new trigonometric identities of the form $2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2} \left(1-\cos\frac{2\pi k}{p}\right)^{p-1-k} = p^{p-2}$ which are derived as a result of relations in a cyclotomic field $\mathcal{R}(\rho)$, where $\mathcal{R}$ is the field of rationals and ρ is a root of unity.

Those identities hold for every positive integer $p \geq 3$ and any proof avoiding cyclotomic fields could be very difficult.

0. Introduction

The trigonometric identities which are being obtained in this paper are a result of ~~rationales~~ RELATIONS in a cyclotomic field $\mathcal{R}(\rho)$, where $\mathcal{R}$ is the field of rationals and ρ is a root of unity. The reader will be familiar with the primitive p-th roots of unity which are the p - 1 different roots of the irreducible polynomial.

$$x^{p-1} + x^{p-2} + \ldots + x + 1 = 0, \qquad p \text{ a prime} > 2 \tag{0.1}$$

which we shall denote by ρ. As is well known

$$\rho = \cos\phi + i\sin\phi, \quad \phi = \frac{2\pi k}{p}, \qquad k = 1, 2, \ldots, k-1 \tag{0.2}$$

Since the p –1 entities $\rho, \rho^2, \ldots, \rho^{p-1}$ form all the different roots of (0.1) we shall choose

$$\rho = \cos\phi + i\sin\phi, \quad \phi = \frac{2\pi}{p}, \quad (k=1) \tag{0.3}$$

The cyclotomic fields have been substantially investigated and the author will make use of some comments.

1. Identities

The following important formula has been proved by Pollard [3] and many other authors and gives the discriminant $D(\rho)$ of the field $\mathcal{R}(\rho)$ in the following explicit form

$$\left.\begin{aligned} D(\rho) &= \prod_{1 \leq i < j \leq p-1} \left(\rho^i - \rho^j\right)^2, \ \rho = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}, \\ D(\rho) &= (-1)^{\frac{p-1}{2}} p^{p-2}, \ \frac{2\pi}{p} = \phi, \ p \text{ prime} > 2, \ \rho = \sqrt[p]{1} \end{aligned}\right\} \tag{1.1}$$

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We shall first investigate

$$T = \prod_{1 \le i < j \le p-1} \left(\rho^i - \rho^j\right); \qquad T^2 = D(\rho) \tag{1.2}$$

We obtain from (1.2)

$$\begin{aligned} T = & \left(\rho - \rho^2\right)\left(\rho - \rho^3\right)\left(\rho - \rho^4\right) \cdots \left(\rho - \rho^{p-1}\right) \\ & \left(\rho^2 - \rho^3\right)\left(\rho^2 - \rho^4\right) \cdots \left(\rho^2 - \rho^{p-1}\right) \\ & \left(\rho^3 - \rho^4\right) \cdots \left(\rho^3 - \rho^{p-1}\right) \\ & \vdots \qquad \left(\rho^{p-2} - \rho^{p-1}\right) \end{aligned}$$

$$= \rho^{p-2}\rho^{2(p-3)}\rho^{3(p-4)}\cdots\rho^{(p-2)}(1-\rho)^{p-2}\left(1-\rho^2\right)^{p-3}\left(1-\rho^3\right)^{p-4}\cdots\left(1-\rho^{p-2}\right)$$

$$= \prod_{k=1}^{p-2} \rho^{(p-1-k)k}\,(1-\rho^k)^{p-k-1} = \prod_{k=1}^{p-2} \rho^{(p-1-k)k} \prod_{k=1}^{p-2} \left(1-\rho^k\right)^{p-k-1}$$

We shall calculate

$$\prod_{k=1}^{p-2} \rho^{(p-1-k)k} = \rho^{\sum_{k=1}^{p-2} (p-k-1)k} \tag{1.3}$$

We have

$$\begin{aligned} \sum_{k=1}^{p-2} (p-k-1)k &= \sum_{k=1}^{p-2} pk - \sum_{k=1}^{p-2} k^2 - \sum_{k=1}^{p-2} k \\ &= \frac{p(p-1)(p-2)}{2} - \frac{(2p-3)(p-1)(p-2)}{2} - \frac{(p-1)(p-2)}{2} \\ &= \frac{p(p-1)(p-2)}{2} - \left(\frac{(2p-3+3)(p-1)(p-2)}{6}\right) \\ &= \frac{p(p-1)(p-2)}{2} - \frac{p(p-1)(p-2)}{3} = \frac{p(p-1)(p-2)}{6} \end{aligned}$$

$$\prod_{k=1}^{p-2} \rho^{(p-1-k)k} = \rho^{\frac{p(p-1)(p-2)}{6}} \tag{1.4}$$

Now

$$\rho = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}\ ; \quad \rho^p = \left(\cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}\right)^p =$$

$$= \cos 2\pi + i \sin 2\pi = 1$$

$$\rho^{\frac{p(p-1)(p-2)}{6}} = \left(\rho^{p}\right)^{\frac{(p-1)(p-2)}{6}}$$

We shall presume here $p > 3$ and shall return to this case later. (In the case $p = 2$, the primitive unity roots are ± 1). We then have $3 \nmid p$, hence $(p-1)(p-2) \equiv 0(6)$ so that either $p-1 \equiv 0(6)$ or $p-2 \equiv 0(3)$ and $p-1 \not\equiv 0(3)$.

Of course, since p is odd, $p-1 \equiv 0(2)$. So any case $6 \mid (p-1)(p-2)$, and since $\rho^{p} = 1$,

$$\rho^{\frac{p(p-1)(p-2)}{6}} = \left(\rho^{p}\right)^{\frac{(p-1)(p-2)}{6}} = 1^{\frac{(p-1)(p-2)}{6}} = 1, \qquad \text{and from (1.4)}$$

$$\prod_{k=1}^{p-2} \rho^{(p-1-k)k} = 1 \tag{1.5}$$

We thus remain with

$$\mathrm{T} = \prod_{k=1}^{p-2} \left(1-\rho^{k}\right)^{p-k-1} \tag{1.6}$$

From (1.6) we obtain, since

$$\rho^{k} = \cos\frac{2\pi k}{p} + i\sin\frac{2\pi k}{p}, \quad \frac{2\pi}{p} = \phi, \quad \frac{2\pi k}{p} = k\phi, \qquad \text{and from (1.6)}$$

$$\mathrm{T} = \prod_{k=1}^{p-2} \left[1-(\cos k\phi + i\sin k\phi)\right]^{p-1-k}$$

$$= \prod_{k=1}^{p-2} \left[(1-\cos k\phi) - i\sin k\phi\right]^{p-1-k}$$

$$= \prod_{k=1}^{p-2} \left(2\sin^{2}\frac{k\phi}{2} - i\,2\sin\frac{k\phi}{2}\cos\frac{k\phi}{2}\right)^{p-1-k}$$

$$= \prod_{k=1}^{p-2} \left(-2\,i\sin\frac{k\phi}{2}\right)^{p-1-k} \left(\cos\frac{k\phi}{2} + i\sin\frac{k\phi}{2}\right)^{p-1-k}$$

$$\mathrm{T} = \prod_{k=1}^{p-2} (-1)^{p-1-k} \prod_{k=1}^{p-2} i^{p-1-k} \prod_{k=1}^{p-2} \left(2\sin\frac{k\phi}{2}\right)^{p-1-k}$$

$$\prod_{k=1}^{p-2} \left(\cos\frac{k\phi}{2} + i\sin\frac{k\phi}{2}\right)^{p-1-k} \tag{1.7}$$

We further have

$\prod_{k=1}^{p-2}(-1)^{p-1-k}=(-1)^{\frac{(p-2)(p-1)}{2}}$ and since p is odd $(-1)^{p-2}=-1$, hence

$$\prod_{k=1}^{p-2}(-1)^{p-1-k}=(-1)^{\frac{p-1}{2}} \tag{1.8}$$

$$\prod_{k=1}^{p-2} i^{p-1-k}=i^{\frac{(p-1)(p-2)}{2}} \tag{1.9}$$

$$\prod_{k=1}^{p-2}\left(\cos\frac{k\phi}{2}+i\sin\frac{k\phi}{2}\right)^{p-1-k}=\prod_{k=1}^{p-2}\left(e^{\frac{i\,k\phi}{2}}\right)^{p-1-k}=$$

$$=\prod_{k=1}^{p-2}e^{\frac{i\phi}{2}(p-1-k)k}=e^{\frac{i\pi}{p}\frac{p(p-1)(p-2)}{6}}=e^{\frac{i\pi\,(p-1)(p-2)}{6}},$$

$$\left.\begin{aligned}&\prod_{k=1}^{p-2}\left(\cos\frac{k\phi}{2}+i\sin\frac{k\phi}{2}\right)^{p-1-k}=(-1)^{\frac{(p-1)(p-2)}{6}},\\ &(p-1)(p-2)\equiv 0(6).\end{aligned}\right\} \tag{1.10}$$

With (1.8), (1.9), (1.10), (1.7) takes the form

$$\mathrm{T}=(-1)^{\frac{p-1}{2}}\, i^{\frac{(p-2)(p-1)}{2}}\,(-1)^{\frac{(p-1)(p-2)}{6}}\prod_{k=i}^{p-2}\left(2\sin\frac{k\phi}{2}\right)^{p-1-k} \tag{1.11}$$

But from (1.2) $D(\rho)=T^2$, so with (p-1) (p-2) $\equiv 0(6)$

$$D(\rho)=(-1)^{p-1}\left(i^2\right)^{\frac{(p-2)(p-1)}{2}}\left((-1)^2\right)^{\frac{(p-1)(p-2)}{6}}\prod_{k=1}^{p-2}2^{p-1-k}\left(2\sin^2\frac{k\phi}{2}\right)^{p-1-k}$$

$$D(\rho)=(-1)^{\frac{(p-1)}{2}}\prod_{k=1}^{p-2}2^{p-1-k}\prod_{k=1}^{p-2}\left(2\sin^2\frac{k\phi}{2}\right)^{p-1-k}$$

$$D(\rho)=(-1)^{\frac{p-1}{2}}\,2^{\frac{(p-2)(p-1)}{2}}\prod_{k=1}^{p-2}(1-\cos k\phi)^{p-1-k}=(-1)^{\frac{p-1}{2}}\,p^{p-2}\quad \text{from (1.1).}$$

Hence we obtain our identity

$$2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2} (1-\cos k\phi)^{p-1-k} = p^{p-2} \tag{1.12}$$

or

$$2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2} \left(1-\cos\frac{2\pi k}{p}\right)^{p-1-k} = p^{p-2} \tag{1.13}$$

It is interesting that the identity (1.13) is valid also for $p = 3$. We have $p = 3$, $k = 1$,

$$2^{\frac{(p-1)(p-2)}{2}} = 2^1 = 2$$

$$2\prod_{k=1}^{1}\left(1-\cos\frac{2\pi}{3}\right)^1 = 2(1-\cos 120°) = 2\left(1+\frac{1}{2}\right) = 3 = 3^{3-2}.$$

But the proof is not valid here, since $3 \mid p$. We find directly, with $\rho = \cos 120° + i \sin 120°$ from (1.1)

$$D(\rho) = \left(\rho-\rho^2\right)^2 = \rho^2\left(1-2\rho+\rho^2\right) = \left(\rho^2-2\rho^3+\rho^4\right),$$

since $\rho^3 = 1$, $(\rho - 1)(\rho^2 + \rho + 1) = 0$, and for $\rho \neq 1$,

$$\rho^2 + \rho + 1 = 0, \rho^2 + \rho = -1.$$

Thus $\rho^2 - 2\rho^3 + \rho^4 = \rho^2 - 2 + \rho = -2 - 1 = -3$,

$$D(\rho) = (-1)^{\frac{3-1}{2}} 3^{3-2} = -3 ,$$

Also
$$\left[\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)-\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)\right]^2 =$$

$$= \left(2i\sin\frac{2\pi}{3}\right)^2 = -4\cdot\frac{3}{4} = -3.$$

Now consider the Vandermonde determinant:

$$V = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \rho & \rho^2 & \cdots & \rho^{p-1} \\ 1 & \rho^2 & \rho^4 & \cdots & \rho^{2(p-1)} \\ . & . & . & \cdots & . \\ 1 & \rho^{p-1} & \rho^{2(p-1)} & \cdots & \rho^{(p-1)^2} \end{vmatrix} =$$

$$= \prod_{0\le i<j<p-1}\left(\rho^i-\rho^j\right) = T\cdot\prod_{j=1}^{p-1}\left(1-\rho^j\right), \quad \text{T as in (1.6).}$$

Multiplying this determinant by itself we obtain

$$V^2 = \begin{vmatrix} p & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & p \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ 0 & 0 & p\cdot & \cdots & 0 \\ 0 & p & 0 & \cdots & 0 \end{vmatrix} = (-1)^{\frac{p-1}{2}} p^p.$$

In view of

$$\sum_{j=0}^{p-1} \rho^{kj} \cdot \rho^j = \sum_{j=0}^{p-1} \rho^{(k+1)j} = \begin{cases} 0 \text{ if } p \nmid k+1 \\ p \text{ if } p \mid k+1 \end{cases}$$

Since $\prod_{j=1}^{p-1} \left(x - \rho^j\right) = 1 + x + x^2 + \ldots + x^{p-1}$, for $x = 1$ we obtain

$\prod_{j=1}^{p-1} \left(1 - \rho^j\right) = p$. Consequently V = pT. Hence $\left|T^2\right| = p^{-2} \left|V^2\right| = p^{p-2}$. From the definition of T we obtain: $\left|T^2\right| = \prod_{k=1}^{p-2} \left(\left|1 - \rho^k\right|^2\right)^{p-k-1}$. But

$\left|1 - \rho^k\right|^2 = 2\left(1 - \cos\frac{2k\pi}{p}\right)$, and then $\left|T^2\right| = \prod_{k=1}^{p-2} \left(2\left(1 - \cos\frac{2k\pi}{p}\right)\right)^{p-1-k}$. Hence (1.13) follows.

Here we did not use the assumption that p is a prime number. Therefore (1.13) holds for every positive integer $p \geq 3$. (1.13) is an interesting identity and any proof avoiding cyclotomic fields could be very difficult.

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Malvina BAICA

The University of Winsconsin
Department of Mathematical
and Computer Sciences
Whitewater, WI 53190, U.S.A.