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ISSN 0327-5647

BULLETIN OF NUMBER THEORY

AND RELATED TOPICS

BOLETIN DE TEORIA DE NUMEROS Y TEMAS CONEXOS

VOL. XVI
1992

UNIVERSIDAD DEL SALVADOR
BUENOS AIRES
ARGENTINA

HILBERT'S DEMAND FOR THE DISCLOSURE OF UNITS IN ALGEBRAIC NUMBER FIELDS

by

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ABSTRACT

Hilbert asked for a universal algorithm by means of which the expansion of any real algebraic number becomes periodic, thus enabling the units of the corresponding algebraic number field to be calculated. In some previous papers the author derived Halter-Koch [3] and Neubrand [2] units using the periodicity of her (ACF) Algorithm. In this paper the author uses the periodicity of the same Algorithm (ACF) over the complex number field [1] to derive Hasse-Bernstein [7] and Halter-Koch and Stender [8] units. With this result the author is able to find all the units in algebraic number fields of the form Q(w) through the action of a unified Algorithm. Thus, the author's Algorithm ACF gives a general framework for finding units in a large number of algebraic number fields and is a significant step toward giving a partial answer to Hilbert's question concerning units.

Classification Number: 11 A 55 and 12 F 05

1. STATEMENT OF THE PROBLEM

In [4] Bernstein proved that the Jacobi-Perron Algorithm (Abbr. JPA) applied to a properly chosen vector in Q(w), $w = \sqrt[n]{D^n + d}$, $d \geq 2$, $d, D \in N$, $d \mid D$, $D \geq (n-2)\, d$ becomes periodic. In [5] he proved that in $Q(\bar{w})$, $\bar{w} = \sqrt[n]{D^n - d}$, $n \geq 2$, $D, d \in N$, $d \mid D$, $D \geq 2\cdot(n-1)\, d$, a properly chosen vector also becomes periodic by the JPA. From these results, Hasse and Bernstein [7] proved that in both cases

$$(1.1) \qquad e_k = \frac{w^k - D^k}{(w-D)^k}, \; k \mid n, \; k > 1$$

$$\bar{e}_k = \frac{\bar{w}^k - D^k}{(\bar{w}-D)^k}, \; k \mid n, \; k > 1$$

comprise $\tau(n) - 1$ units in the corresponding fields $Q(w)$ and $Q(\bar{w})$.

The shortcomings of these very important results are the restriction on d and the bounds on D. Later Halter-Koch and Stender [8] widened the range for d in order to obtain units in Q(w), but they did not use a general algorithm, in proving that the expressions

$$(1.2) \qquad e_k = \frac{w^k - D^k}{(w-D)^k} \quad \text{are units if } d \mid D^n.$$

In this paper we will obtain Hasse Bernstein units (1.1) and Halter-Koch and Stender units (1.2) as particular cases of the author's units from ACF where the restrictions on d and D in both cases can be removed. Only the requirement that $d \mid D$ remains.

2. INTRODUCTORY FORMULATIONS

We shall briefly describe the background which leads to our new results. We start with

$$(2.1) \quad w = \sqrt[n]{D^n + d}, \; d \mid D, \; D \in N, \; d \in Z, \; n \geq Z.$$

From (2.1) we obtain

$$(2.2) \qquad w^n - D^n = d.$$

Hence

$$(2.3) \qquad (w - D_1)(w - D_2) \ldots (w - D_n) - d = 0$$

where we use the notation

$$D_k = \rho^k D,\ \rho = e^{2\pi i/n},\ k = 1, \ldots, n.$$

From the theory of the n^{th}–roots of unity, ρ^k, $1 \le k \le n$, these numbers form a multiplicative group and $\rho\rho^2 \ldots \rho^n = 1$. Then the above mentioned D_k, $1 \le k \le n$ are all different. Then also different are the linear factors in (2.3) with the notations

$$(2.4) \qquad f_{i,k}(w) = \prod_{s=i}^{k} (w - D_s)$$

$$f_{i,i}(w) = w - D_i,\ 1 \le i \le k \le n$$

we then construct the fixed starting vector

$$(2.5) \qquad a^{(0)} = (f_{1,n-1}(w), f_{1,n-2}(w), \ldots, f_{1,2}(w), f_{2,2}(w))$$

where the first $(n-2)$ components, but not the last component, contain the linear factor $w - D_1$. We rearrange the D_k in the sense that $\{D_1, D_2, \ldots, D_n\} = \{D, \rho D, \rho^2 D, \ldots, \rho^{n-1} D\}$, so that $D_k = \rho^i D$, $k, i = 1, 2, \ldots, n$. Let $\{D_1, D_2, \ldots, D_n\}$ be a fixed permutation of $\{D, \rho D, \rho^2 D, \ldots, \rho^{n-1} D\}$.

Then we carry out the ACF of $a^{(0)}$ by the rule

$$(2.6) \qquad b_i^{(v)} = a_i^{(v)}(D_1),\ i = 1, \ldots, n-1;\ v = 0, 1, \ldots$$

Applying (2.6) to (2.5) we obtain the companion vector since

$$a_i^{(0)}(D_1) = 0,\ i = 1, \ldots, n-2;\ a_{n-1}^{(0)}(D) = D_1 - D_2.$$

$$(2.7) \qquad b^{(0)} = (0, 0, \ldots, 0, D_1 - D_2),$$

and from here we obtain the vector

$$a^{(1)} = \left(\frac{f_{1, n-2}(w)\ f_{n, n}(w)}{d}, \frac{f_{1, n-3}(w)\ f_{n, n}(w)}{d}, \ldots \right.$$

$$\left. \ldots, \frac{f_{1, 1}(w)\ f_{(n, n)}(w)}{d}, \frac{f_{n, n}(w)}{d} \right).$$

since $f_{1, n}(w) = d$.

The main point is that in all further steps obtaining $a^{(v)}$, $a_i^{(v)}$, $(i = 1, \ldots, n-2)$ contains the linear factor $w - D_1$, but $a_{n-1}^{(v)}$ does not contain that factor, that is, $b_i^{(v)} = 0$ $(i = 1, \ldots, n-2)$ and $b_{n-1}^{(v)} = D_1 - D_k$ or $\frac{D_1 - D_k}{d}$, $k \neq 1$, leading finally to periodicity of the algorithm with $a^{(0)} = a^{(n-1)n}$ for $d \neq 1$. The $n(n-1)$ vectors of the primitive period consists of n cycles of vectors each containing $n - 1$ vectors. With the exception of order of d^{-1}, these cycles are all equal. For $s = 0, 1, \ldots, n-1$, each cycle leads to the product

$$d^{-1} a_{n-1}^{(s(n-1))} a_{n-1}^{(s(n-1)+1)} \ldots a_{n-1}^{(s(n-1)+n-2)},$$

where only one cycle does not have the factor d^{-1}. The components of the $b^{(v)}$ are all zeroes, with the exception of $b_{n-1}^{(v)}$ which has the form $\frac{D_1 - D_k}{d}$ or $D_1 - D_i$, so that the components of the companion vectors are all algebraic integers in view of $d | D_m$, $m = 1, \ldots, n$.

For all of this work the polynomial $P(x) = x^n - D^n - d$ was considered for the field equation. Its irreducibility over the field of rationals was proved by the author in her paper [1]. To obtain Hasse-Bernstein and Halter-Koch and Stender units as particular cases of Baica's units from the periodicity of her ACF Algorithm we will use another polynomial to be discussed in the next section.

3. THE MAIN RESULT

We introduce

$$
(3.1)\qquad \begin{cases} P(x) = \left(\prod_{i=1}^{k} (x^{s_i} - D_i^{s_i}) \right) - d; \\ k \geq 2,\ s_i \geq 1;\ D_i \in N;\ d|D_i; \\ d \in z;\ i = 1, 2, \ldots, k;\ |d| \geq 1; \\ 0 < D_1 < D_2 < \ldots < D_k. \end{cases}
$$

We shall now prove a series of theorems concerning the polynomial D(x) as defined in (3.1)

THEOREM 1. The polynomial P(x) as defined in (3.1) is irreducible over the field of rationals in infinitely many cases.

Proof. We first prove

LEMMA 1. Let

$$
(3.2)\qquad \begin{cases} F(x) = \left(\prod_{i=1}^{k} (x^{s_i} - t_i) \right) - d; \\ k \geq 2,\ s_i \geq 1;\ t_i, d \in Z;\ d|t_i \\ (i = 1, 2, \ldots, k);\ |d| > 1;\ d \text{ square free.} \end{cases}
$$

Then F(x) is irreducible over the field of rationals.

To prove Lemma 1 we first recall Eisenstein's irreducibility criterion. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$, $a_i \in R$, $i = 0, 1, \ldots, n$; let p be a prime such that $p|a_i$, $i = 1, \ldots, n$; $p^2 \nmid a^n$. Then f(x) is irreducible over the field of rationals. If we multiply out the factors in F(x) we obtain

$$
F(x) = x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n
$$

$n = s_1 + s_2 + \ldots + s_k \geq 2$. We have, by condition of F(x),

$d|b_i$, $i = 1, \ldots, n$.

$b_n = t_1 \cdot t_2 \cdot t_3 \ldots t_k - d$.

Now, since $d|t_i$ $(i = 1, \ldots, k)$ and $k \geq 2$, we have, since $p|d$, $p^q|t_1\ t_2 \ldots t_k$, hence

$p^2|t_1 t_2 \ldots t_k$. If $p^2|b_n$, then $p^2|d$, against presumption. This proves Lemma 1.

Now if we set in P(x); $D_i^{s_i} = t_i$ ($i = 1, \ldots, k$), $|d| > 1$, d square free, then the conditions of Lemma 1 are fulfilled and thus Theorem 1 is proved for $|d| \neq 1$. For instance, the polynomial $(x-4)(x^3-8^3)(x^5-64^5)-2$ is irreducible, but without further investigations nothing could be said about the irreducibility of the polynomial $(x^3-15^3)(x^5-40^5)(x^8-55^8)-25$ or the polynomial $(x-4)(x^3-8^3)(x^5-64^5)-1$. Theorem 1 remains valid if the conditions of P(x) are replaced by $D_i \in Z$, and the magnitude ordering of Di is dropped. The magnitude ordering of D_i, $0 < D_1 < D_2 < \ldots < D_k$, was introduced in (3.1) for convenience.

The reader should also note that without the restriction $|d| > 1$, Lemma 1 could not be applied to prove Theorem 1.

Now let prove the irreducibility of P(x) for the case $|d| = 1$ also. Since Eisenstein's divisibility criterion does not work in this case, we must appeal to the magnitude criterion of Bernstein [6] p. 73-78. Bernstein proves that:

THEOREM 2. Let the polynomial F(x) be

$$(3.3)\quad \begin{cases} F(x) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \ldots + k_{n-1} x - d \\ d, k_j \ (j = 1, \ldots, n-1) \text{ rational integers} \\ d \neq 0;\ d|k_j;\ |k_{n-1}| \geq c|d| (2 + B), \\ B = \sum_{i=0}^{n-2} |k_i|,\ k_0 = 1;\ c \geq 1. \end{cases}$$

Then f(x) has one, and only one real root, w, which lies in the interval

$$(3.4)\quad \begin{cases} 0 < w < \dfrac{2}{B+4} \text{ for } \dfrac{k_{n-1}}{d} > 0 \\ \dfrac{-2}{B+4} < w < 0 \text{ for } \dfrac{kn-1}{d} < 0. \end{cases}$$

The reader will note that all k_j, $j = 1, \ldots, n-2$, can vanish but $|k_{n-1}| \geq 3\ c|d|$. Bernstein also proves [6] p. 78.

THEOREM 3. The polynomial (3.3) with $|k_{n-1}| \geq 2\ |d|B$, $B \geq 2$ is irreducible over Q.

We rearrange the polynomial (3.3) as

$$(3.5)\quad \begin{cases} x^n - D^n - d = [(x-D) + D]^n - D^n - d \\ = \sum\limits_{i=0}^{n-1} \binom{n}{i} D^i (x-D)^{n-i} - d. \end{cases}$$

Our variable from (3.5) is now $x - D$ and $k_i = \binom{n}{i} D^i$, $i = 1, \ldots, n-1$.

We stipulate

$$|k_{n-1}| = \binom{n}{n-1} D^{n-1} \geq 2|d| \cdot 2^n (1 + D + D^2 + \ldots + D^{n+2})$$

$$= 2^{n+1} |d| \frac{D^{n-1} - 1}{D - 1},$$

and with

$$(3.6)\quad \begin{cases} D \geq 2^{n+1} \\ k_{n-1} = n D^{n-1} \geq \dfrac{2^{n+1} |d| D^{n-1}}{\frac{1}{2} D} \end{cases}$$

$$(3.7)\quad nD \geq 2^{n+2} |d|.$$

Thus we have

THEOREM 4. The number $w = \sqrt[n]{D^n + d}$, $n \geq 2|d|$, $d|D$, $D \in N$, $d \in Z$, $D \geq 2^{n+1}$ is an irrational integer of degree n.

Proof. We obtain $n \geq 2|d|$ from (3.6) and (3.7) stipulating $nD \geq n\, 2^{n+1} \geq 2^{n+2}|d|$. The conditions for B in Theorems 2 and 3 are fulfilled, and thus Theorem 4 is proved. Therefore $x^n - D^n - d$ has a real root and is irreducible over Q for $|d| \geq 1$.

In the above proof the case $|d| = 1$ is included. From $n \geq 2\ |d|$ it follows that for $|d| = 1$, $n \geq 2$, as it should be. The estimate $D \geq 2^{n+1}$ is an inexact approximation. For example we assume that $\sqrt[n]{2^n + 1}$ is an n–th degree irrational with $D \geq 1$, $|d| = 1$ and $D \ngeq 2^{n+1}$. The use of Bernstein's irreducibility Theorem is a magnitude criterion, and not a divisibility criterion. In his book [6], p. 82-87, he proves an irreducibility theorem where the condition $d | k_i$ $(i = 1, \ldots, n-1)$, k_i from Theorem 2 is dropped.

We now prove the main result of this paper.

THEOREM 5. The polynomial P(x) from (3.1) has at least one real root $\overline{w}$ if

(1) $d > 0$ $(d \geq 2)$

(2) $d < 0$ $(d \leq -2)$; $k = 1 \pmod 2$

(3) $d < 0$ $(d \leq -2)$; $D_k - D_{k-1} \geq 2|d|$.

Proof.

(1) Here $P(D_1) = -d < 0$;

$P(2D_k) \geq (2D_k - D_1)\, D_k - d > 0$.

Thus P(x) changes signs at least once between D1 and $2D_k \neq D_1$.

(2) Here $P(D_1) = -d > 0$.

$P(0) = -d + (-1)^k D_1^{s_1} D_2^{s_2} \ldots D_k^{s_k}$, and

since now $k = 3, 5, \ldots$, $P(0) \leq -D_1 D_2 D_3 - d$.

But $D_1 D_2 D_3 \geq |d|^3$; hence

$P(0) < -|d|^3 - d = -|d|^3 + |d| < 0$.

Thus P(x) changes signs at least once between 0 and $D_1 > 0$.

(3) We have $D_1 \geq |d|; D_2 \geq |d|, \ldots, D_{k-1} \geq (k-1)\,|d|$,

$D_k \geq k\,|d|$.

We have again $P(D_1) = -d > 0$. Let $x = D_k + d$.

Since $D_k - D_{k-1} \geq 2\,|d|$, $D_i < D_k$, $i = 1, \ldots, k-1$, $D_k - D_i \geq 2\,|d|$.

Further

$$\prod_{i=1}^{k-1} (x^{s_i} - D_i^{s_i}) = m \prod_{i=1}^{k-1} (x - D_i),$$

$m \in N$, $m \geq 1$. Thus

$$\prod_{i=1}^{k-1} (x^{s_i} - D_i^{s_i}) = m \prod_{i=1}^{k-1} (D_k + d - D_i) \geq$$

$$\geq m \prod_{i=1}^{k-1} (2\,|d| + d) \geq m\,|d|.$$

Thus

$$P(x) = \left[\prod_{i=1}^{k} (x^{s_i} - D_i^{s_i})\right] - d =$$

$$= \left[\prod_{i=1}^{k-1} (x^{s_i} - D_i^{s_i})\right] (x^{s_k} - D_k^{s_k}) - d,$$

$$= \left[\prod_{i=1}^{k-1} (x^{s_i} - D_i^{s_i})\right] (x - D_k)\; t - d,$$

$t \in N$, $t \geq 1$. Hence, since $x = D_k + d$,

$$P(x) = d\,t \prod_{i=1}^{k-1} (x^{s_i} - D_i^{s_i}) - d, \text{ and since } d < 0;\ 1 \leq t,$$

$$\prod_{i=1}^{k-1} (x^{s_i} - D_i^{s_i}) \geq m\,|d|, m \in N, m \geq 1, |d| \geq 2,$$

$$P(x) \leq d\,t\,m\,|d| - d < 0.$$

Thus $P(x)$ changes signs between D_1 and $D_k + d \neq D_1$, which proves (3) and thus Theorem 5 is proved.

We did not succeed in eliminating the condition (3) attached to Theorem 5. That this condition (3) is only sufficient, is shown by the following example:

Let $-d = D_1 = \frac{1}{2} D_2 = D \in N;\ D \geq 2,$

$$P(x) = (x - D)\,(x - 2D) + d.$$

This parabola cuts the x–axis in $x_{1,2} = \frac{1}{2}(3D \pm \sqrt{D^2 - 4D})$. Here $D_k - D_{k-1} = 2D - D = D < 2\,|d|$, but $P(x)$ is still irreducible if D is square free. We have obtained the significant result that $P(x)$, defined by (3.1) with the restriction (3) of Theorem 5 is irreducible in infinitely many cases, and has at least one real root in each of these cases. Denoting

$$(3.8)\qquad \begin{cases} s_1 + s_2 + \ldots + s_k = n, \\ s_i \text{ from } (3.1) \end{cases}$$

we see that $\bar{w}$, being a root of the irreducible $P(x)$ from (3.1) which satisfies condition (3) of Theorem 5, is an n – th degree irrational. If $P(x)$ is formed as stated in (3.1), it can have at most n real irrational roots. $P(x)$ cannot have repeated roots nor rational roots in the above case.

We now factor $P(x)$ from (3.1) and introduce the notation:

$$(3.9)\qquad P(x) = \prod_{j=1}^{k} (x - D_j)\ (x - \rho_j D_j) \ldots (x - \rho_j^{\,s_j - 1} D_j).$$

$$(3.10)\qquad \begin{cases} P(\bar{w}) = 0;\ \rho_j = e^{\frac{2\pi_i}{s_j}}; \\ \{\ldots, D_j, \rho_j D_j, \rho^2{}_j D_j, \ldots, \rho^{s_j - 1} D_j, \ldots\} \\ = \{\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_n\}\ ;\ j = 1, \ldots, k. \end{cases}$$

From (3.10) we see that any number of the first set of (3.10) can equal any number of its second set. We shall construct an ACF involving the numbers $\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_n$, and once a pairing of these numbers with those of the first set of (3.10) has been fixed, we must retain this choice during the process of applying the (ACF) Algorithm. We shall verify that the number of the two sets in (3.10) are all different so that these sets are indeed well defined. For let be

$$\rho_u{}^a D_u = \rho_v{}^b D_v;\ u, v = 1, \ldots, k.$$
$$a = 0, 1, \ldots, s_u - 1;\ b = 0, 1, \ldots, s_v - 1.$$

We obtain $|\rho_u{}^a D_u| = |\rho_v{}^b D_v| \Rightarrow Du = Dv$, against presumption. Since we shall need the fact that the sets in (3.10) are well defined, the ordering of the D_i in (3.1) is more plausible. We shall operate with an (ACF) on the numbers $\overline{w}, \overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n, P(\overline{w}) = 0, \overline{D}_i$ ($i = 1, \ldots, n$) from (3.10). Let the l.c.m. of $s_1, s_2, \ldots, s_k$ be $\overline{m} = [s_1, s_2, \ldots, s_k]$, $\overline{\rho} = e^{\frac{2\pi_i}{m}}$; $Q(\overline{w}, \overline{D}_1, \overline{D}_2, \ldots, \overline{D}_n) = Q(\overline{w}, \overline{\rho})$, where $Q(\overline{w}, D_1) = = Q(\overline{w}, D_2) = \ldots = Q(\overline{w}, D_k) = Q(\overline{w})$. We shall now construct a fixed vector $\overline{a}^{(0)} \in Q(\overline{w}, \overline{\rho})$ and then proceed with an ACF in complete analogy with the ACF of $a^{(0)}$.

Let construct the starting vector

$$
(3.11) \qquad \begin{cases} \overline{a}^{(0)} = (f_{1,n-1}(\overline{w}), f_{1,n-2}(\overline{w}), \ldots, f_{1,2}(\overline{w}), f_{2,2}(\overline{w})) \\ f_{i,k}(\overline{w}) = \prod_{s=i}^{k} (\overline{w} - \overline{D}_s), \\ f_{i,i}(\overline{w}) = \overline{w} - \overline{D}i, \ 1 \le i \le k \le n \\ P(\overline{w}) = 0; P(x) \text{ from (3.1)} \\ \text{and irreducible in infinitely many cases.} \end{cases}
$$

For the generation of the companion vectors we use the formula

$$
(3.12) \qquad \overline{b}_i^{(v)} = \overline{a}_i^{(v)}(D_1);\ i = 1, \ldots, n-1,\ v = 0, 1 \ldots .
$$

THEOREM 6. The (ACF) of $\overline{a}^{(0)}$ from (3.11) with the generating formula (3.12) for the companion vectors is purely periodic and the length of the primitive period of the (ACF) equals $m = n(n-1)$ for $d \neq 1$ and $m = n-1$ for $d = 1$. Apart from slight changes in notation, the proof of Theorem 6 follows those of Theorem 1 and Theorem 2 of [1] verbatim. We further obtain, as in the case of the ACF for $a^{(0)}$.

Corollary 1 to Theorem 6.

The product of the $n-1$–st components of the $n(n-1)$ vectors of the primitive period of the (ACF) of $\overline{a}^{(0)}$ equals

$$
(3.13) \qquad d^{-(n-1)} ((\overline{w} - D_2)(\overline{w} - D_3) \ldots (\overline{w} - D_n))^n.
$$

In the same way we obtain

Corollary 2 to Theorem 6.

The components of the n(n–1) companion vectors of (ACF) of $\bar{a}^{(0)}$ equal

$$(3.14)\quad \begin{cases} \bar{b}_i^{(v)} = 0,\ i = 1, \ldots, n-2;\ v = 0, 1, \ldots, n(n-1) \\ \bar{b}_{n-1}^{(v)} = \bar{D}_1 - \bar{D}_i,\ i = 2, \ldots, n \text{ or} \\ \bar{b}_{n-1}^{(v)} = d^{-1}\ (\bar{D}_1 - \bar{D}_i),\ i = 2, \ldots, n. \end{cases}$$

Thus all these companion vectors (3.14) are algebraic integers.

We now turn to get units in the field $Q(\bar{w}, \bar{\rho})$ of type K

We obtain Theorem 7 from Corollaries 1 and 2 to Theorem 6.

THEOREM 7. A unit in the field $Q(\bar{w}, \bar{\rho})$ is given by the expression (3.13), viz.,

$$(3.15)\quad \bar{e} = e^{-(n-1)}\ (\ (\bar{w} - \bar{D}_2)\ (\bar{w} - \bar{D}_3)\ \ldots\ (\bar{w} - \bar{D}_n)\ )^n.$$

Since $\quad (\bar{w} - \bar{D}_1)\ (\bar{w} - \bar{D}_1)\ \ldots\ (\bar{w} - \bar{D}_n) = d,.$

$$(3.16)\quad \frac{(\ (\bar{w} - \bar{D}_1)\ (\bar{w} - \bar{D}_2)\ \ldots\ (\bar{w} - \bar{D}_n)\ )^n}{d^n} = 1$$

so that the expression on the left side of (3.16) is equal 1, and is, therefore a unit. Dividing this expression by the expression (3.13) we obtain:

$$(3.17)\quad \bar{e}_1 = \frac{(\bar{w} - \bar{D}_1)^n}{d} \quad \text{is a unit in } Q\ (\bar{w}, \bar{\rho}).$$

Since $\bar{D}_1$ could be any number out of the n numbers $\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_n$, we have generally (changing each time the fixed starting vector $\bar{a}^{(0)}$ and the generating rule for the companion vector of the ACF of $\bar{a}^{(0)}$ accordingly):

$$(3.18)\quad \begin{cases} \bar{e}_t = (\bar{w} - \bar{D}_t)^n,\ t = 1, \ldots, n, \\ \text{are units in } Q\ (\bar{w}, \bar{\rho}). \end{cases}$$

Choosing for $\bar{D}_t = D_1, D_2, \ldots, D_k$ (from P(x) in (3.1)), we obtain the important result

$$(3.19)\qquad \begin{cases} \bar{e}_i = \dfrac{(\bar{w} - D_i)^n}{d} \quad i = 1, 2, \ldots, k, \\ \text{are units in } Q(\bar{w}). \end{cases}$$

Thus the k units (3.19) are units in the real algebraic number field $Q(\bar{w})$ of degree n. (3.19) are the Hasse-Bernstein and Halter-Koch and Stender units with the only restriction d|D.

From (ACF) we obtain more units and now we choose the s_i units in $Q(\bar{w}, \bar{\rho}_i)$,

$$(3.20)\qquad \begin{cases} \bar{e}_{i,0} = \dfrac{(\bar{w} - D_i)^n}{d}, \\ \bar{e}_{i,1} = \dfrac{(\bar{w} - \rho_i D_i)^n}{d}, \\ \bar{e}_{i,2} = \dfrac{(\bar{w} - \rho_i{}^2 D_i)^n}{d}, \ldots, \\ \bar{e}_{i,s_i-1} = \dfrac{(\bar{w} - \rho_i{}^{s_i-1} D_i)^n}{d}. \end{cases}$$

In (3.20) only the units $\bar{e}_{i,0}$ are Hasse-Bernstein and Halter-Koch and Stender units. If we multiply all the units of (3.20) by each other we obtain the unit

$$\bar{\bar{e}}_i = \frac{((\bar{w} - D_i)(\bar{w} - \rho_i D_i) \ldots (\bar{w} - \rho_i{}^{s_i-1} D_i))^n}{d^{s_i}}$$

$$(3.21)\qquad \bar{\bar{e}}_i = \frac{(\bar{w}^{s_i} - D_i{}^{s_i})^n}{d^{s_i}} \quad (i = 1, \ldots, k) \text{ are k units in } Q(\bar{w}).$$

If we raise (3.19) to the power s_i and divide by (3.21)

$$(3.22) \qquad e_i^* = \left(\frac{(\bar{w} - D_i)^{s_i}}{\bar{w}^{s_i} - D_i^{s_i}} \right)^n, (i = 1, \ldots, k) \text{ are units in } Q(\bar{w}).$$

Therefore, since $\sqrt[n]{e_i^*}$ are units in $Q(\bar{w})$, we obtain that

$$(3.23) \qquad \bar{e}_i^* = \frac{(\bar{w} - D_i)^{s_i}}{\bar{w}^{s_i} - D_i^{s_i}} \ ;(i = 1, \ldots, k) \text{ are also units in } Q(\bar{w}).$$

For $s_i = 2$ we obtain $\bar{e}_2^* = \dfrac{\bar{w} - D_i}{\bar{w} + D_i}$. Here we shall terminate our

investigation of units in $Q(\bar{w}, \bar{p})$ and in the real algebraic field $Q(\bar{w})$. We must stress that only the units (3.15) were originally obtained by an (ACF); all the others were first obtained by simple algebraic number theoretic considerations. Including these results and some previous results in other papers the author was able to obtain all of the known units and other new units in the algebraic number fields of type $Q(w)$ from a common algorithm (ACF).

The conclusion is that even with those significant results we are still some distance from the realization of Hilbert's daring dream to find the full group of fundamental units of any algebraic number field by the means of an universal algorithm.

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